A Geometric Approach to Revealed Preference via Hamiltonian Cycles

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Abstract

It is shown that a fundamental question of revealed preference theory, namely whether the Weak Axiom of Revealed Preference (WARP) implies the Strong Axiom (SARP), can be reduced to a Hamiltonian cycle problem: A set of bundles allows a preference cycle of irreducible length if and only if the convex monotonic hull of these bundles admits a Hamiltonian cycle. This leads to a new proof to show that preference cycles can be of arbitrary length for more than two but not for two commodities. For this it is shown that a set of bundles satisfying the given condition exists if and only if the dimension of the commodity space is at least three. Preference cycles can be constructed by embedding a cyclic $(L-1)$-polytope into a facet of a convex monotonic hull in L-space, because cyclic polytopes always admit Hamiltonian cycles. An immediate corollary is that WARP only implies SARP for two commodities. The proof is intuitively appealing as is gives a geometric interpretation of preference cycles.

Keywords: Cyclic Polytopes;Hamiltonian Cycles; Preference Cycles; Revealed Preference; Strong Axiom of Revealed Preference; Weak Axiom of Revealed Preference

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1 Introduction

This article shows that within the standard framework of demand theory a revealed preference cycle of a certain length that does not contain cycles of shorter length must admit a Hamiltonian cycle on the convex monotonic hull of all commodity bundles involved in the cycle. Conversely, if the convex hull of a set of bundles does admit a Hamiltonian cycle, then there exist corresponding budget sets such that these bundles form a preference cycle. Hamiltonian cycles on convex hulls in the commodity space thus characterise revealed preference cycles, and therefore also violations of utility maximising demand behaviour. This result can be used to answer old questions in a new, geometric, intuitive way. It also highlights a connection between David Gale's work on mathematics and revealed preference.

Based on the work by, among many others, Afriat (1967) and Varian (1982, 1983), revealed preference has become operational and offers useful tools for the analysis of consumption data, which is also exemplified by the growing trend to collect consumption data in experimental settings[1]. These data can now be easily tested for consistency with revealed preference axioms, and thereby also for consistency with utility maximisation and other hypotheses. But for quite some time it had been an open question in economic theory whether the Weak Axiom of Revealed Preference (WARP) as introduced by Samuelson (1938) was actually sufficient to guarantee that a demand function maximises a utility function. Houthakker (1950) defined an apparently stronger condition, the Strong Axiom of Revealed Preference (SARP) and showed that this condition was indeed sufficient.

Arrow (1959), however, remarked that there was still no proof "that the Weak Axiom is not sufficient to ensure the desired result. The question is still open." Uzawa (1959) showed that the Weak Axiom combined with certain regularity conditions implies the Strong Axiom.[2] Meanwhile, Rose (1958) showed that the Weak Axiom implies the Strong Axiom for two commodities, extending a geometrical argument by Hicks (1965 [1956], pp. 52–54).

Finally, Gale (1960) constructed a counterexample for the case of three commodities: WARP was satisfied, SARP was violated. This, essentially, settled the question. Kihlstrom et al. (1976) provided a theoretical argument which yields an infinite number of demand functions that satisfy WARP but not SARP. Peters and Wakker (1994, 1996) showed how to embed Gale's example in higher dimensions without relying on isomorphic extensions, that is, with strictly positive demand for every commodity for suitable budgets. John (1997) showed that there is a simpler proof of their results.

Samuelson (1953) raised the question whether the exclusion of cycles of a certain length would be sufficient to imply SARP. Even if WARP does not

[1]See for example Sippel 1997, Mattei 2000, Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007).

[2]Samuelson is said to have expressed the view that these regularity conditions "are perhaps integrability conditions in disguise" (Gale, 1960), and Kihlstrom et al. (1976) commented that "it looks very much like the *strong axiom* itself".

generally imply SARP, the exclusion of cycles of length K could exclude the possibility of cycles of length greater than K. This question was answered by Shafer (1977) who showed that, in the three dimensional case, for every positive integer K, there exists a demand function which violates SARP, but has no cycles involving K or fewer observations. For $K = 2$ this also proves that WARP does not imply SARP. Shafer's result was also extended into more than three dimensions by Peters and Wakker (1994) and John (1997). In a recent paper, Deb and Pai (2012) examine the set of preferences than can be observed, depending on the dimension of the commodity space. They find that if the dimension of the commodity space is high enough relative to the number of observations, any revealed preference relation can arise.

This paper develops a new geometric and intuitive approach to show that with more than two commodities there can be preference cycles of arbitrary finite length whereas for two commodities cycles can only be of length 2. From this it immediately follows (i) that WARP necessarily implies SARP for two commodities, (ii) that there exist finite sets of observations which satisfy WARP but not SARP. The approach here is an alternative to the proofs of Rose, Gale, and Peters and Wakker insofar as it gives a condition which is both necessary and sufficient for the existence of cycles of length greater than two. It is then shown that the condition cannot be satisfied in two dimensions, whereas in more than two dimensions it can be satisfied. The proof technique is intuitively appealing: the condition states that for a set of K bundles one can find a set of K budget sets on which these bundles are chosen such that these observations form a preference cycle of irreducible length K if and only if the convex monotonic hull of these bundles admits a Hamiltonian cycle involving all K bundles. Thus, it admits an understanding by giving a geometric interpretation of preference cycles. Finite but otherwise arbitrarily long preference cycles in $L > 2$ dimensions can then be constructed by embedding a cyclic $(L-1)$-polytope into an $(L-1)$-face of a convex monotonic hull in L-space. Cyclic polytopes always admit Hamiltonian cycles and have been described and analysed by David Gale (1963) three years after he provided his famous example of a preference cycle in three dimensions. To the best of the author's knowledge, Gale himself never mentioned this connection in his works.

2 Preliminaries

We use the following notation: $\mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x \geqq \mathbf{0}\}$, $\mathbb{R}^L_{++} = \{x \in \mathbb{R}^L : x > \mathbf{0}\}$, where " $x \geqq y$ " means " $x_i \geq y_i$ for all i ", " $x \geq y$ " means " $x \geqq y$ and $x \neq y$ ", " $x > y$ " means " $x_i > y_i$ for all i ", and $\mathbf{0}$ is the null vector. Note the convention to use subscripts to denote scalars or vector components and superscripts to index bundles.

Let $X = \mathbb{R}^L_+$ be the commodity space, where $L \geq 2$ denotes the number of different commodities. Budget sets are of the form $B(p) = \{x \in X : px \leq 1\}$ for some price vector $p \in \mathbb{R}^L_{++}$.

The demand function $D : \mathbb{R}^L_{++} \to X$ of a consumer assigns to each budget set the commodity bundle chosen by the consumer. Demand is exhaustive (i.e., $p\,x = 1$). Denote the boundary of the budget set $B(p)$ as $\partial B(p) = \{x \in X : px = 1\}$, so $D(p) \in \partial B(p)$. A binary relation $\succsim$ on X *represents* D if for every budget set $B(p)$ we have $D(p) = x$ with $x \succsim y$ for all $y \in B(p)$.

We assume to have a finite set of $N \geq 2$ observations indexed by $i \in \{1, \ldots, N\}$. An observation is a pair (x, p) where $x = D(p)$. The set of observations can then be denoted $S = \{(x^i, p^i)\}_{i=1}^N$. We also denote $B(p^i) = B^i$ and $\partial B(p^i) = \partial B^i$.

Let R^0 be a binary relation, the directly revealed preference relation, on X. A chosen bundle x^i is *directly revealed preferred* to x, written $x^i R^0 x$, if $p^i x^i \geq p^i x$. It is *revealed preferred* to x, written $x^i R^* x$, if for some sequence of bundles $(x^j, x^k, \ldots, x^m)$ it is the case that $x^i R^0 x^j$, $x^j R^0 x^k$, …, $x^m R^0 x$. In this case R^* is the *transitive closure* of the relation R^0.

The set S satisfies WARP if whenever $x^i R^0 x^j$, $x^i \neq x^j$, $\{i, j\} \subseteq \{1, \ldots, N\}$, we have [not $x^j R^0 x^i$]. It satisfies SARP if whenever $x^i R^* x^j$, $x^i \neq x^j$, $\{i, j\} \subseteq \{1, \ldots, N\}$, we have [not $x^j R^* x^i$].

The set S can be interpreted as an unweighted directed graph (digraph), i.e. a pair $G = (V, A)$ where V is the set of nodes (the observations) and A is the set of directed arcs (the directly revealed preference relations). An arc $a_{ij} = \{(x^i, p^i), (x^j, p^j)\}$ is directed from (x^i, p^i) to (x^j, p^j) and is an element of A if $x^i R^0 x^j$.[3]

Let $\mathcal{I} = \{\mathcal{I}(1), \ldots, \mathcal{I}(K)\}$ be a set of K indices. A set $\{(x^i, p^i)\}_{i \in \mathcal{I}}$ of K observations forms a *revealed preference cycle of length* K if $x^{\mathcal{I}(k)} R^0 x^{\mathcal{I}([k \mod K]+1)}$ for $k \in \{1, \ldots, K\}$. For example, if $\mathcal{I} = \{1, 2, 3, 4, 5\}$, then $\{(x^i, p^i)\}_{i \in \mathcal{I}}$ forms a revealed preference cycle of length 5 if $x^1 R^0 x^2 R^0 x^3 R^0 x^4 R^0 x^5 R^0 x^1$. A set $\{(x^i, p^i)\}_{i \in \mathcal{I}}$ forms a *revealed preference cycle of irreducible length* K if there does not exist a proper subset $\mathcal{I}' \subset \mathcal{I}$ with K' indices such that $\{(x^i, p^i)\}_{i \in \mathcal{I}'}$ forms a revealed preference cycle of length K'. As an illustration, suppose there is a set of observations $\{(x^i, p^i)\}_{i \in \mathcal{I}}$, $\mathcal{I} = \{1, \ldots, 5\}$ such that $x^1 R^0 x^2 R^0 x^3 R^0 x^4 R^0 x^5 R^0 x^1$. Suppose that also $x^2 R^0 x^4$, but there are no other directly revealed preference relations. Then $\{(x^i, p^i)\}_{i \in \mathcal{I}'}$ with $\mathcal{I}' = \{1, 2, 4, 5\}$ forms a revealed preference cycle of irreducible length 4. Thus, $\{(x^i, p^i)\}_{i \in \mathcal{I}}$ does *not* form a revealed preference cycle of irreducible length 5. See Figure 1.

Obviously, WARP implies the absence of cycles of irreducible length two in the set S, whereas SARP implies the absence of cycles of any irreducible length in the set S (i.e., with N observations, SARP implies the absence of cycles of length N).

The *convex hull* CH of a set of points $Y = \{y^i\}_{i=1}^M \subset X$ is defined as

$$\mathrm{CH}(Y) = \left\{ x \in X : x = \sum_{i=1}^M \lambda_i\, y^i, y^i \in Y, \lambda_i \geq 0, \sum_{i=1}^M \lambda_i = 1 \right\}.$$

[3] See also Wakker (1989), who takes a graph theoretic approach to revealed preference and defines alternatives as vertices and revealed preference relations as arcs. He shows that a choice function satisfies congruency (a condition "similar" yet not equivalent to SARP, cf. Richter 1966) if and only if all dicircuits (directed paths that form a cycle) are reversible.

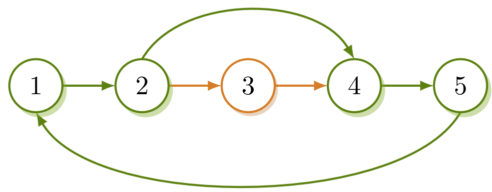


Figure 1: The observations can be interpreted as nodes of a digraph. The shortest cycle includes nodes 1, 2, 4 and 5.

The *convex monotonic hull* of a set of points Y will be denoted as

$$\mathrm{CMH}(Y) = \mathrm{CH}\big(\{x \in X : x \geq y^i \text{ for some } i \in \{1, \dots, M\}\}\big),$$

and $int\mathrm{CH}(Y)$ and $int\mathrm{CMH}(Y)$ denote the interior of $\mathrm{CH}(Y)$ and $\mathrm{CMH}(Y)$, while $\partial\mathrm{CH}(Y) = \mathrm{CH}(Y)\backslash int\mathrm{CH}(Y)$ and $\partial\mathrm{CMH}(Y) = \mathrm{CMH}(Y)\backslash int\mathrm{CMH}(Y)$ denote the boundaries.

The convex monotonic hull will be a central part of the proofs in the next section, so it is worthwhile to offer some economic interpretation. This interpretation is not required for the formal proofs, but it motivates the use of it. Varian (1982) shows how to construct the set $\mathrm{RP}(x^0)$ of all bundles which are revealed preferred to some bundle x^0 (which is not necessarily observed as a choice). This set $\mathrm{RP}(x^0)$ is the set of all bundles which must have a higher utility value for every monotonic concave utility function which is maximised by the observed choices, given the budget constraints. That is, any bundle $x \in \mathrm{RP}(x^0)$ can only be demanded at budgets which, if observed, would lead to xR^*x^0. Varian (1982) and Knoblauch (1992) show that

$$int\mathrm{CMH}(\{x^i : x^i R^* x^0\}) \subseteq \mathrm{RP}(x^0) \subseteq \mathrm{CMH}(\{x^i : x^i R^* x^0\}).$$

The observation of a preference cycle, and thus a violation of SARP, can be interpreted as a form of irrationality of the consumer, because SARP tests for a strictly concave and monotone utility function which generates the data (see Matzkin and Richter 1991). However, taking revealed preference literally, a preference cycle implies that the consumer is indifferent between every bundle in the cycle. Thus it is not surprising that if we observe such a cycle, all the bundles will be on the closure of $\mathrm{RP}(x^0)$.

The following definitions are based on Grünbaum (2003) and Brøndsted (1983). An *affine subspace* of $\mathbb{R}^L$ is a translate of a linear subspace, that is, a subset $A = x + \mathcal{L}$, $x \in \mathbb{R}^L$ and $\mathcal{L}$ is a linear subspace of $\mathbb{R}^L$. For a set $H \subset \mathbb{R}^L$, the intersection of all affine subspaces containing H is the *affine hull* of H.

A *convex polytope* (or simply *polytope*) $\mathbf{P}_L$ in $\mathbb{R}^L$ is a set which is the convex hull of a non-empty finite set of elements in $\mathbb{R}^L$. A point $x^i \in \mathbf{P}_L$ is an *extreme point* of $\mathbf{P}_L$ if $x^i = \lambda\, x^j + (1-\lambda)x^k$, $\lambda \in (0,1)$, x^j, $x^k \in \mathbf{P}_L$ imply that $x^i = x^j = x^k$. A convex polytope is the convex hull of all its extreme points (Brøndsted 1983, Theorem 5.10). A *supporting hyperplane* H of $\mathbf{P}_L$ is a hyperplane that contains at least point of $\mathbf{P}_L$ and $\mathbf{P}_L$ is contained in one of the two closed half-spaces determined by H. A set $F \subset \mathbf{P}_L$ is a *proper face* (or simply *face*) of $\mathbf{P}_L$ if there exists a supporting hyperplane H of $\mathbf{P}_L$ such that $F = \mathbf{P}_L \cap H$, $F \neq \emptyset$, and $F \neq \mathbf{P}_L$.

The dimension of a face F is the dimension of the affine hull of F. A face F is called a $k-$face if the dimension of F is k. A $0-$face is called a *vertex*, and a $1-$face is called an *edge*. An $(L-1)-$face is also called a *facet*. Then an equivalent definition of an extreme point is to say that $x \in \mathbf{P}_L$ is an extreme point if $\{x\}$ is a vertex. Strictly speaking, a vertex is a set containing a single element. However we will also call an element x a vertex if $\{x\}$ is a $0-$face. Two distinct vertices of $\mathbf{P}_L$ are called *adjacent* if the line segment joining them is an edge of $\mathbf{P}_L$. Two faces of $\mathbf{P}_L$ are *incident* if one contains the other. Thus, a vertex and an edge are incident if the vertex is a vertex of the edge.

Following Klee (1965, 1966), a *path* in $\mathbf{P}_L$ is a sequence x^1, x^2, …, x^K of consecutively adjacent vertices of $\mathbf{P}_L$, and K is the length of the path. A path is *simple* (or of irreducible length) if no vertex is repeated. A path is a *simple cycle* (or circuit) if $n \geq 2$ and $x^1 = x^n$. A *Hamiltonian cycle* on $\mathbf{P}_L$ is a cycle which visits each vertex of $\mathbf{P}_L$ once and only once, except for the initial vertex which is visited twice. A simple cycle is not a simple path, as the initial vertex is identical to the last vertex. As paths and cycles are defined in terms of vertices, an element on the boundary of $\mathbf{P}_L$ that is not an extreme point (that is, a vertex) cannot be an element of a path or cycle.

In the specific application we have in mind, we may have a set T of vertices of a $\mathbf{P}_L$, with $x^i = x^j$, $i \neq j$, and both $x^i \in T$ and $x^j \in T$. Then if all other $x^k \in T \backslash \{x^i, x^j\}$ are distinct and $x^k \neq x^i$, we could either say that a cycle involving all elements of T is of length $K-1$ if T consists of K elements, or we could say that a cycle involving all elements of T is not Hamiltonian. Both conventions are possible for the purpose of the paper. We chose the latter and say that a Hamiltonian cycle is defined as a cycle involving distinct elements of $\mathbf{P}_L$.

So far, we have defined Hamiltonian cycles on convex polytopes. However, we need to extend this definition to cycles on convex monotonic hull, which strictly speaking are not polytopes. This generalisation is straightforward to define: The definitions of faces carries over to convex monotonic hulls. Therefore, paths, simple paths, and cycles (including Hamiltonian cycles) can be defined on convex monotonic hulls as above.

Cyclic polytopes are defined using the following procedure (Gale 1963, Klee 1966): The *moment curve* $\mathcal{M}_L$ in $\mathbb{R}^L$ is the curve parameterised by $f(t) = (t, t^2, \ldots, t^L)$ for $t \in \mathbb{R}_+$, where $2, \ldots, L$ are exponents, not indices. That is, the moment curve $\mathcal{M}_L$ is the subset of $\mathbb{R}^L$ which consists of all points of the form $(t, t^2, \ldots, t^L)$. Two polytopes are *combinatorially equivalent* (or *isomorphic*) if there is a one-to-one correspondence between their faces such that incidence and dimensions are preserved. A *cyclic polytope* is a polytope which is combinatorially equivalent to a polytope which is the convex hull of $m \geq L+1$ distinct points $\{f(t_1), \ldots, f(t_m)\}$ on $\mathcal{M}_L$ (i.e., all vertices of the polytope are on $\mathcal{M}_L$).

Cyclic polytopes therefore exist in all $\mathbb{R}^L$ by construction. Klee (1966, Theorem 1.1) showed that a cyclic polytope always admits a Hamiltonian cycle. Thus, there always exist polytopes which admit Hamiltonian cycles. Perhaps not coincidentally, cyclic polytopes were described and analysed by Gale (1963), who also was the first to construct an example of a preference cycle in three dimensions of irreducible length greater than two, thereby showing that WARP does not imply SARP.

3 Results

Consider the following casual observation: There can be observed choices that are in the interior of a set $\mathrm{RP}(x)$ and hence are redundant for the construction of $\mathrm{RP}(x)$ (that is, they are not vertices of $\mathrm{RP}(x)$). If an observed choice x^i is directly preferred to such an interior point, the budget hyperplane ∂B^i has to intersect the set $\mathrm{RP}(x)$. Then ∂B^i has at least one vertex of $\mathrm{RP}(x)$ on its "left" side, so x^i is also directly revealed preferred to at least one other vertex of $\mathrm{RP}(x)$. This leads to the following theorem.

Theorem 1 *Let $T = \{x^j\}_{j\in\mathcal{J}}$ be a set of commodity bundles, with $\mathcal{J}$ a set of $K \geq 3$ distinct indices. The following conditions are equivalent:*

1. *There exists a set of price vectors $\{p^j\}_{j\in\mathcal{J}}$ and a permutation $\mathcal{I}$ of $\mathcal{J}$ such that $\{(x^i, p^i)\}_{i\in\mathcal{I}}$ forms a revealed preference cycle of irreducible length K.*
2. *The convex monotonic hull of all elements of T, $\mathrm{CMH}(T)$, admits a Hamiltonian cycle involving all elements of T.*

All proofs are in the appendix.

We can now use Theorem 1 to prove that we cannot find any set of bundles for the two commodity case to construct a preference cycle.

Proposition 1 *For the case of commodity space $\mathbb{R}^2_+$ there cannot be revealed preference cycles of irreducible length greater than two.*

See also Figure 2 for an illustration.

As a simply corrollary, we note the following:

Corollary 1 *For the case of commodity space $\mathbb{R}^2_+$ WARP implies SARP for any finite set S of data.*

Corollary 1 follows directly from Theorem 1 and Proposition 1.

This concludes the two commodity case. We now turn to the case of more than two commodities. It will be shown that the conditions of Theorem 1 can always be satisfied.

Proposition 2 *For the case of commodity space $\mathbb{R}^L_+$, $L > 2$ the shortest revealed preference cycle can be of arbitrary irreducible length.*

The proof of the proposition (Section A.3) is based on embedding a cyclic $(L-1)-$polytope into a higher dimensional space. This is done by embedding

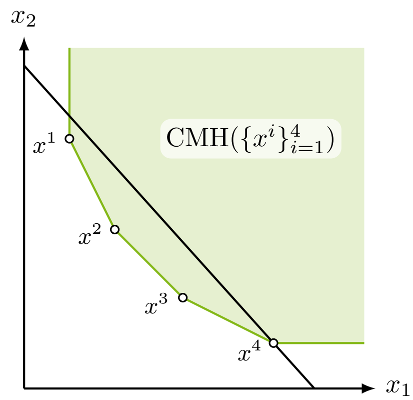


Figure 2: The observation x^4 cannot be directly revealed preferred to x^1 without also being directly revealed preferred to x^2 and x^3.

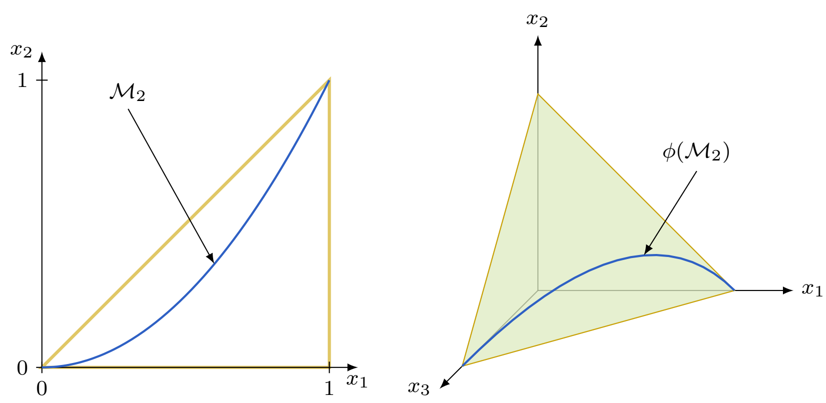
the moment curve $\mathcal{M}_{L-1}$ (see Figure 3.a) into an $(L-1)$-simplex in $\mathbb{R}^L_+$ (see Figure 3.b).

Now follows as a corollary that WARP does not imply SARP.

Corollary 2 *For the case of commodity space $\mathbb{R}^L_+$, $L > 2$, there exist finite sets of observations which satisfy WARP but not SARP.*

Corollary 2 follows directly from Theorem 1 and Proposition 2.

Remark 1 *Note that we can only embed a $\mathbf{P}_1$ polytope into the unit simplex in $\mathbb{R}^2$. A $\mathbf{P}_1$ consists of only two vertices. The fact that WARP does not imply SARP in more than two dimensions can also be shown by letting T be the set of vertices of an embedded $(L-1)$-simplex, but this only allows to construct cycles of irreducible length L.*



(a) (b)

Figure 3: (a) The moment curve $\mathcal{M}_2$ is contained in the 2−simplex given by the convex hull of $(0,0)$, $(0,1)$, and $(1,1)$. (b) The unit 2−simplex in $\mathbb{R}^3_+$ and the moment curve $\mathcal{M}_2$.

A Appendix

A.1 Proof of Theorem 1

Theorem 1 Without loss of generality, we let $\mathcal{I} = \{1, \dots, K\}$ to simplify the exposition of the proof.

(1) $\Rightarrow$ (2) Assume that the conditions in (1) are satisfied. We need to show that

(i) all $x^i \in T$ are distinct vertices on CMH(T), and
(ii) x^{i-1} is adjacent to x^i for all $i \in \mathcal{I}$ and x^K is adjacent to x^1.

(i) If x^i is not a distinct vertex, then either
(i.1) x^i is not distinct (i.e., $x^i = x^j$ for some $i \neq j$), or
(i.2) x^i is not a vertex because $x^i \in \partial\text{CMH}(T)\backslash\text{CH}(T)$ or $x^i \in int\text{CMH}(T)$, or
(i.3) x^i is not a vertex because $x^i \in \partial\text{CH}(T)\backslash\text{CMH}(T)$.

(i.1) Suppose all elements of T are vertices but not all are distinct, i.e. $x^i = x^j$ for some $j \neq i$. Then $\{x^i, x^j\} \subset B^i$, which implies $x^i R^0 x^j$, and $\{x^i, x^j\} \subset B^j$, which implies $x^j R^0 x^i$. Thus, the set indexed by $\mathcal{I}' = \{i, j\}$ has a revealed preference cycle of length $2 < K$, contradicting (1).

(i.2) Suppose $x^i \in T$ is not a vertex because either $x^i \in \partial\text{CMH}(T)\backslash\text{CH}(T)$ (a point on the boundary of CMH(T) but on the monotonic extension of CH(T)) or $x^i \in intB^i$. In both cases, there exists an $x \in \partial\text{CH}(T)$ such that $x^i \geq x$ (i.e. $x^i_j \geq x_j$ and $x^i \neq x^j$). Then $x \in intB^i$. But by (1), we have $x^{i-1}R^0x^i$ and therefore $x^i \in B^{i-1}$, and therefore $x \in intB^{i-1}$.

Suppose first that $x = x^j$ for some $x^j \in T$. Then $x^i R^0 x^j$, and because $x^j \in B^{i-1}$, is is also true that $x^{i-1}R^0x^j$. Then if $j < i-1$ there exists a set $\mathcal{I}' \subset \mathcal{I}$ with $\mathcal{I}' = \{j, j+1, j+2, \dots, i-2, i-1\}$ with $x^j R^0 x^{j+1} \dots R^0 x^{i-1}$, and with $x^{i-1}R^0x^j$, thus the set indexed by $\mathcal{I}'$ forms a revealed preference cycle of length $K' = K-1$, contradicting (1). If $j > i$, then the set indexed by $\mathcal{I}' = \{i-1, j, j+1, \dots, K-1, K, 1, \dots, j-1, j\}$ forms a revealed preference cycle of length $K' = K-1$, again contradicting (1).

So suppose $x \notin T$. As $x \in \partial\text{CH}(T)$ it can be expressed as a linear combination of at least two elements of $T\backslash\{x^i\}$ which are on the boundary of CH(T). Suppose x is a linear combination of $\{x^j\}_{j\in\mathcal{K}}$ with $\mathcal{K} \subset \mathcal{I}$, $i \notin \mathcal{K}$, and $x^j \in \partial\text{CH}(T)$ for all $j \in \mathcal{K}$. Then $\text{CH}(\{x^j\}_{j\in\mathcal{K}}) \subseteq \partial\text{CH}(T)$. As $x^j \in B^{i-1}$ implies $x^{i-1}R^0x^j$, in order to have a revealed preference cycle of length K at most one element of $\{x^j\}_{j\in\mathcal{K}}$ can be in B^{i-1}. If there is an $x^j \in B^{i-1}$, $j \in \mathcal{K}$, then $j = i-1$. As $i \notin \mathcal{K}$, there must be at least one $j \in \mathcal{K}$ with $j \neq i-1$.

Suppose $i - 1 \notin \mathcal{K}$. Then $x^j \in B^{i+1}$ for all $j \in \mathcal{K}$. As ∂B^{i-1} separates $\mathbb{R}^L$ into two half-spaces, all $\{x^j\}_{j\in\mathcal{K}}$ are in the same half-space with respect to ∂B^{i-1}. As both $\text{CH}(\{x^j\}_{j\in K_1})$ and a half-space are convex sets and $x \in B^{i-1}$ and $x^j \notin B^{i-1}$, $\text{CH}(\{x^j\}_{j\in\mathcal{K}})$ is contained in the half-space opposite of x. But

$x \in \mathrm{CH}(\{x^j\}_{j\in\mathcal{K}})$, a contradiction. Thus for some $j \in \mathcal{K}$, $x^j \in B^{i-1}$, and therefore $x^{i-1}R^0x^j$ with $j \neq i$. Then there exists a set of indices $\mathcal{I}' \subset \mathcal{I}$ with $i \notin \mathcal{I}'$ such that the set indexed by $\mathcal{I}'$ forms a revealed preference cycle of length $K' \leq K-1$, contradicting (1).

Suppose $i-1 \in \mathcal{K}$. As we have assumed that demand is exhaustive, $x^{i-1} \in \partial B^{i-1}$. This time, $x^j \notin B^{i-1}$ for all $j \in \mathcal{K}\backslash\{i-1\}$ for the same reasons as before. Then $\mathrm{CH}(\{x^j\}_{j\in\mathcal{K}}) \cap B^{i-1} = \{x^{i-1}\}$, but $x \in \mathrm{CH}(\{x^j\}_{j\in\mathcal{K}})$ and $x \notin B^{i-1}$, a contradiction.

(i.3) Suppose $x^i \in T$ is not a vertex because while it is on the boundary of $\mathrm{CH}(T)$ and not on the monotonic extension, it is not an extreme point. Then x^i can be expressed as a linear combination of at least two elements of $T\backslash\{x^i\}$ which are on the boundary of $\mathrm{CH}(T)$. Then we can apply the same arguments as in part (i.2).

This completes the proof that all $x^i \in T$ are distinct vertices on $\mathrm{CMH}(T)$.

(ii) Suppose the line segment connecting x^{i-1} and $x^i \in T$ (the edge joining $\{x^{i-1}\}$ and $\{x^i\}$) is not on the boundary of $\mathrm{CMH}(T)$, with $x^i \neq x^j$. With $\{x^{i-1}, x^i\} \subset B^{i-1}$ and convexity of B^{i-1}, every $x = \lambda\, x^{i-1} + (1-\lambda)x^i$ for some $\lambda \in (0,1)$ is in B^{i-1}. At least one such x must be in the interior of $\mathrm{CMH}(T)$ given that the line segment connecting x^{i-1} and x^i is not an edge. Thus, there must be some $\tilde{x} \leq x$ on the boundary of $\mathrm{CMH}(T)$, with $\tilde{x} \in B^{i-1}$. As $\lambda \in (0,1)$ and $x^i \neq x^j$, $x \neq x^i$ and $x \neq x^j$.

Suppose $\tilde{x} = x^j$ implies $j \notin \{i-1, i\}$. Then $x^j \in B^{i-1}$ and therefore $x^{i-1}R^0x^j$. there exists a set of indices $\mathcal{I}' \subset \mathcal{I}$ with $i \notin \mathcal{I}'$ such that the set indexed by $\mathcal{I}'$ forms a revealed preference cycle of length $K' \leq K-1$, contradicting (1). Suppose instead $\tilde{x} \notin T$; then $\tilde{x}$ is a linear combination of at least two elements of $T\backslash\{x^i\}$. At least one of those elements must be in B^{i-1}, because otherwise $\tilde{x} \leq x$ cannot be a linear combination of them. Say, this element is x^j; then again, $x^j \in B^{i-1}$ and the same results as before follows.

Suppose the line segment connecting x^{i-1} and $x^i \in T$ is on the boundary of $\mathrm{CMH}(T)$, but intersects with the line segment connecting x^{j-1} and $x^j \in T$ for some $j \neq i$. Then $x = \lambda\, x^{i-1} + (1-\lambda)x^i = \mu\, x^{j-1} + (1-\mu)x^j$ for some $\lambda, \mu \in (0,1)$ is the intersection of these two line segments. Then a similar argument as in the preceding paragraph applies (i.e., one of x^{j-1} or x^j must be in B^{i-1}, which results in a subset with a revealed preference cycle of length less than K).

Thus, the line segments connecting x^{i-1} and x^i and x^{j-1} and x^j do not intersect and are edges of $\mathrm{CMH}(T)$. Then x^{i-1} and x^i are adjacent, and the sequence x^1, x^2, …, x^K, x^1 is a Hamiltonian cycle, which concludes this part of the proof.

(2) ⇒ (1) Suppose the conditions in (2) are satisfied. By definition of vertices and Hamiltonian cycles, if $\mathrm{CMH}(T)$ admits a Hamiltonian cycle involving all elements of T, then all elements of T are vertices on $\mathrm{CMH}(T)$. Furthermore, if the sequence of vertices which constitutes the cycle is x^1, x^2, …, x^K, x^1, then by definition x^{i-1} and x^i are adjacent for all $i \in \{2, 3, \ldots, K, 1\}$. By the supporting hyperplane theorem, there exists a hyperplane $H(q) = \{y \in X : qy = a\}$ such that $x^{i-1} \in H(q)$ and $x^i \in H(q)$ for some $q \in \mathbb{R}^L$ (where q and a depend on x^{i-1} and x^i).

By the conditions (2) the line segment connecting x^{i-1} and x^i is an edge on CMH(T); thus, this line segment is a $1-$face on CMH(T). Let $F_{i,j}$ be that face. By the definition of faces in Section 2, there exists a supporting hyperplane H' such that $F_{i,j} = \text{CMH}(T) \cap H'$. Thus, there must exist a $q \in \mathbb{R}^L$ such that $\{x^{i-1}, x^i\} \subset H(q)$ and $F_{i,j} = H(q) \cap \text{CMH}(T)$.

Next we show that a can be chosen to be 1. If $a = 0$, then $\mathbf{0} \in H(q)$. But then $x^{i-1} \in H(q)$ and $x^i \in H(q)$ implies that they are on a ray through the origin. Then either $x^{i-1} \leq x^i$ or $x^i \leq x^{i-1}$, but then either x^i or x^{i-1} is not a vertex of CMH(T). Thus $a \neq 0$ and $a = 1$ is possible by normalising q.

Next we show that each q can be chosen such that $q > \mathbf{0}$. Clearly, if $a = 1$, at least one q_j, $j \in \{1, \ldots, L\}$ must be strictly positive because all elements of T are in $\mathbb{R}^L_+$. Now suppose that for all supporting hyperplanes, $q_k \leq 0$ for at least one $k \in \{1, \ldots, L\}$. Then there is a $y \in \mathbb{R}^L_{++}$ with $y \geq x^{i-1}$ and $y \in H(q)$. But then y is not a linear combination of x^{i-1} and x^i, and therefore $y \notin F_{i,j}$. Then $H(q)$ is not the supporting hyperplane used to define the edge $F_{i,j}$. But there exists a supporting hyperplane $H(q')$ for some q' with $q'_j > 0$ such that $F_{i,j} = H(q') \cap \text{CMH}(T)$, contradicting the assumption that for all supporting hyperplanes, $q_k \leq 0$ for at least one $k \in \{1, \ldots, L\}$.

Thus if $a = 1$, we must have $q > \mathbf{0}$.

Let q be the price vector at which x^{i-1} was chosen, so that $p^i = q$, $\partial B^{i-1} = \partial B(q) = H(q)$. As $H(q) \cap \text{CMH}(T)$ is the edge connecting x^{i-1} and x^i, we have $x^i \in \partial B^{i-1}$ and $x^j \notin B^{i-1}$ for all $j \notin \{i-1, i\}$. Then $x^{i-1} R^0 x^i$ and [not $x^{i-1} R^0 x^j$] for all $j \notin \{i-1, i\}$. Thus we can find budget hyperplanes for each $i \in \mathcal{I}$ such that $\{(x^i, p^i)\}_{i \in \mathcal{I}}$ forms a preference cycle of irreducible length K. ■

A.2 Proof of Proposition 1

Proposition 1 Let $T = \{x^i\}_{i \in \mathcal{I}}$, with $\mathcal{I} = \{1, \ldots, K\}$, $K > 2$, be a set of bundles, such that each $x^i \in T$ is a distinct vertex on CMH(T). Let $a = \arg\max_i \{x^i_1\}_{i \in \mathcal{I}}$, $b = \arg\max_i \{x^i_2\}_{i \in \mathcal{I}}$, $y = (x^a_1 + 1, x^a_2)$, and $z = (x^b_1, x^b_2 + 1)$. Any $\mathbf{P}_2$ with K distinct vertices has at most K edges, and every vertex is incident to at most two edges and is adjacent to at most two vertices (see Brøndsted 1983, Theorems 10.5 and 12.16). In the convex polytope CH($\{x^i\}_{i \in \mathcal{I}} \cup \{y, z\}$), x^a is adjacent to y and x^b is adjacent to z. Thus, on CH($\{x^i\}_{i=1}^K \cup \{y, z\}$), x^a and x^b can each only be adjacent to one of the elements in T. With CH($\{x^i\}_{i \in \mathcal{I}} \cup \{y, z\}$) $\subset$ CMH($\{x^i\}_{i \in \mathcal{I}}$) it follows that on CMH($\{x^i\}_{i \in \mathcal{I}}$) both x^a and x^b can only be adjacent to one other element of T.

We can count the total number of adjacencies: If x^{i-1} and x^i are adjacent, this is one and only one adjacency; thus, if there are K distinct vertices of a $\mathbf{P}_2$, and each vertex is adjacent to two other vertices, then the total number of adjacencies is K. As x^a and x^b are adjacent to only one other vertex, the total number of adjacencies on $\mathrm{CMH}(\{x^i\}_{i\in\mathcal{I}})$ among the K elements of T can therefore only be at most $(K-1)$, which is one less than is needed for a Hamiltonian cycle. Thus, the conditions of Theorem 1 cannot be satisfied. See also Figure 2 for an illustration. ■

A.3 Proof of Proposition 2

Proposition 2 It needs to be shown that a set of bundles $T = \{x^i\}_{i\in\mathcal{I}}$, $T \subset \mathbb{R}^L_+$ which satisfies the conditions in Theorem 1 always exists. We will do so by embedding a cyclic polytope $\mathbf{P}_{L-1}$ in $\mathbb{R}^L_+$. Cyclic polytopes always admit Hamiltonian cycles (1966, Theorem 1.1).

Let $\mathbf{A}$ be an $L \times (L-1)$ matrix, defined as $\mathbf{A}_{i,j} = 0$ if $i > j$ and $\mathbf{A}_{i,j} = 1$ if $i \le j$, that is,

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

Let $\mathbf{a}_j$ denote the jth row. Let $\tilde{\Delta}^{L-1} = \mathrm{CH}(\{\mathbf{a}_j\}_{j=1}^L)$; $\tilde{\Delta}^{L-1}$ is an $(L-1)$-simplex in $\mathbb{R}^{L-1}$. For $y \in \tilde{\Delta}^{L-1}$, let $\phi(y) = (1-y_1, y_1-y_2, \dots, y_{L-2}-y_{L-1}, y_{L-1})$. Then $y = \sum_{j=1}^L \phi(y)_j \mathbf{a}_j$, that is, every $y \in \tilde{\Delta}^{L-1}$ can be expressed as a linear combination of the row vectors $\mathbf{a}_j$.

Let Δ^{L-1} be the unit $(L-1)$-simplex in $\mathbb{R}^L$, that is, the convex hull of the unit vectors in $\mathbb{R}^L$. Let $\mathbf{e}_j$ be the jth unit vector in $\mathbb{R}^L$, that is, $\mathbf{e}_1 = (1,0,\dots)$, $\mathbf{e}_2 = (0,1,0,\dots)$, ...

The previously defined ϕ is a bijection $\phi : \tilde{\Delta}^{L-1} \to \Delta^L$. For all $y \in \tilde{\Delta}^{L-1}$, we have

$$\begin{aligned}\phi(y) &= \phi\left(\sum_{j=1}^L \phi(y)_j \mathbf{a}_j\right) \\ &= \sum_{j=1}^L \phi(y)_j \mathbf{e}_j.\end{aligned}$$

That is, every element $y \in \tilde{\Delta}^{L-1}$ can be expressed as a linear combination of the $\mathbf{a}_j$s and corresponds to an element $x \in \Delta^{L-1}$, $x = \phi(y)$, which can be expressed as a linear combination of the unit vectors using the same scalars. By construction, for all sets $\{y^1, y^2, \dots, y^M\} \subset \tilde{\Delta}^{L-1}$ and for all $\lambda \in [0,1]^M$ with

$\sum_{j=1}^{M} \lambda_j = 1$, we have

$$\phi\left(\sum_{j=1}^{M} \lambda_j y^j\right) = \sum_{j=1}^{M} \lambda_j \phi(y^j),$$

that is, ϕ is an affine map from $\tilde{\Delta}^{L-1}$ to Δ^{L-1}.

Recall the definition of $f(t)$, the function that parameterises the moment curve $\mathcal{M}_{L-1}$: $f(t) = (t, t^2, \ldots, t^{L-1})$, where $1, \ldots, L-1$ are exponents. Then with $t \in [0,1]$ all $f(t) \in [0,1]^{L-1}$, and all $f(t)$ are contained in $\tilde{\Delta}^{L-1}$ (see Figure 3.(a) for an illustration). Therefore also the convex hull of all points or of a subset of points from the moment curve are contained in $\tilde{\Delta}^{L-1}$. Consider a set $T' = \{f(t_1), \ldots, f(t_K)\}$ of K points from $\mathcal{M}_{L-1}$, such that CH(T') is a cyclic $\mathbf{P}_{L-1}$. Let $T = \{\phi(f(t_1)), \ldots, \phi(f(t_K))\}$; then clearly CH($T$) is a polytope which is combinatorially equivalent to the cyclic polytope CH(T'), as all adjacencies and the dimension of faces are preserved by the affine map ϕ: If $F' \subset$ CH(T') is a k-face of CH(T'), then $F = \{x \in \mathbb{R}^L_+ : \exists y \in F', x = \phi(y)\}$ is a k-face of CH(T). See Figure 3.(b) for an illustration.

Let H be the hyperplane in $\mathbb{R}^L$ which contains Δ^{L-1} (i.e., the affine hull of Δ^{L-1}). CH(T) is the cyclic polytope in $\mathbb{R}^L_+$ which we just constructed; we have CH(T) $\subset H$. Then H is a supporting hyperplane of CMH(T), and CH(T) is a facet of CMH(T) (i.e. an $(L-1)$-face). By construction, all $x \in T$ are distinct vertices on CH(T), thus also on CMH(T), and all edges of CH(T) are edges on CMH(T). Thus, CMH(T) allows a Hamiltonian cycle involving all K vertices of CH(T) and therefore by Theorem 1 a preference cycle of irreducible length K exists.

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References

Afriat, S. N. (1967): The Construction of Utility Functions From Expenditure Data, *International Economic Review*, 8(1), 67-77.

Andreoni, J. and J. Miller (2002): Giving According to GARP: An Experimental Test of the Consistency of Preferences for Altruism, *Econometrica*, 70(2), 737-753.

Arrow, K. J. (1959): Rational Choice Functions and Orderings, *Economica*, 26(102), 121-127.

Brøndsted, A. (1983): *An Introduction to Convex Polytopes*, volume 90 of *Graduate Texts in Mathematics*, Springer-Verlag New York.

Choi, S., R. Fisman, D. Gale, and S. Kariv (2007): Consistency and Heterogeneity of Individual Behavior under Uncertainty, *American Economic Review*, 97(5), 1921-1938.

Deb, R. and M. M. Pai (2012): The Geometry of Revealed Preference, *working paper*

Fisman, R., S. Kariv, and D. Markovits (2007): Individual Preferences for Giving, *American Economic Review*, 97(5), 1858-1876.

Gale, D. (1960): A Note on Revealed Preference, *Economica*, 27(108), 348-354.

——— (1963): Neighborly and Cyclic Polytopes, in: *Convexity: Proceedings of Symposia in Pure Mathematics*, American Mathematical Society, 225-232.

Grünbaum, B. (2003): *Convex Polytopes*, volume 221 of *Graduate Texts in Mathematics*, Springer-Verlag New York.

Harbaugh, W. T., K. Krause, and T. R. Berry (2001): GARP for Kids: On the Development of Rational Choice Behavior, *American Economic Review*, 91(5), 1539-1545.

Hicks, J. R. (1965 [1956]): *A Revision of Demand Theory*, Oxford: Oxford University Press.

Houthakker, H. S. (1950): Revealed Preference and the Utility Function, *Economica*, 17(66), 159-174.

John, R. (1997): A Simple Cycle Preserving Extension of a Demand Function, *Journal of Economic Theory*, 72(2), 442-445.

Kihlstrom, R., A. Mas-Colell, and H. Sonnenschein (1976): The Demand Theory of the Weak Axiom of Revealed Preference, *Econometrica*, 44(5), 971-978.

Klee, V. (1965): Paths on Polyhedra. I, *Journal of the Society for Industrial and Applied Mathematics*, 13(4), 946-956.

——— (1966): Paths on Polyhedra. II, *Pacific Journal of Mathematics*, 17(2), 249-262.

Knoblauch, V. (1992): A Tight Upper Bound on the Money Metric Utility Function, *American Economic Review*, 82(3), 660-663.

Mattei, A. (2000): Full-scale real tests of consumer behavior using experimental data, *Journal of Economic Behavior & Organization*, 43(4), 487-497.

Matzkin, R. and M. K. Richter (1991): Testing Strictly Concave Rationality, *Journal of Economic Theory*, 53(2), 287-303.

Peters, H. and P. Wakker (1994): WARP Does Not Imply SARP for More Than Two Commodities, *Journal of Economic Theory*, 62(1), 152-160.

——— (1996): Cycle-Preserving Extension of Demand Functions to New Commodities, *Journal of Mathematical Economics*, 25(3), 281-290.

Richter, M. K. (1966): Revealed Preference Theory, *Econometrica*, 34(3), 635-645.

Rose, H. (1958): Consistency of Preference: The Two-Commodity Case, *Review of Economic Studies*, 25(2), 124-125.

Samuelson, P. A. (1938): A Note on the Pure Theory of Consumer's Behavior, *Economica*, 5(17), 61-71.

——— (1953): Consumption Theorems in Terms of Overcompensation rather than Indifference Comparisons, *Economica*, 20(77), 1-9.

Shafer, W. J. (1977): Revealed Preference Cycles and the Slutsky Matrix, *Journal of Economic Theory*, 16, 293-309.

Sippel, R. (1997): An Experiment on the Pure Theory of Consumer's Behavior, *The Economic Journal*, 107(444), 1431-1444.

Uzawa, H. (1959): Preference and Rational Choice in the Theory of Consumption, in: K. J. Arrow, S. Karlin, and P. Suppes (Editors), *Mathematical Methods in the Social Sciences*, Stanford, California: Stanford University Press, 129-148.

Varian, H. R. (1982): The Nonparametric Approach to Demand Analysis, *Econometrica*, 50(4), 945-972.

——— (1983): Non-parametric Tests of Consumer Behaviour, *Review of Economic Studies*, 50(1), 99-110.

Wakker, P. (1989): A Graph-Theoretic Approach to Revealed Preference, *Methodology and Science*, 22, 53-66.