Quasiconcave Preferences on the Probability Simplex: A Nonparametric Analysis

Jan Heufer*

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*TU Dortmund University, Department of Economics and Social Science, Chair of Microeconomic Theory, Vogelpothsweg 87, D-44227 Dortmund, Germany. Email: jan.heufer@tu-dortmund.de

Abstract

A nonparametric approach is presented to test whether decisions on a probability simplex could be induced by quasiconcave preferences. Necessary and sufficient conditions are presented which can easily be tested. If the answer is affirmative, the methods developed here allow to reconstruct bounds on indifference curves. Furthermore we can construct quasiconcave utility functions in analogy to the utility function constructed in the proof of Afriat's Theorem. The approach is of interest for ex-ante fairness considerations when a dictator is asked to choose probabilities to win an indivisible prize. It is also of interest for decisions under risk and stochastic choice. It allows nonparametric interpersonal comparisons.

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1 Introduction

Suppose an individual can choose a point on a subset of a probability simplex that represents probabilities of different consumers for winning a prize. An individual (the dictator in an experimental setting) can give up some of his own probability of winning the prize in exchange for a fairer ex ante allocation. Such preferences for fairness have been considered by Karni and Safra (2002a) and Karni and Safra (2002b) and experimentally investigated by Karni et al. (2008). The theoretical analysis implies that individuals with preferences for fairness have quasiconcave preferences in the probabilities; the experimental analysis indicates that this is often the case.

A related yet different topic are certain deviations from the expected utility (EU) hypothesis. EU implies that an individual's indifference curves in a probability space are straight parallel lines. Empirical evidence, however, shows that this is generally not the case. Allais and Hagen (1979), Kahneman and Tversky (1979), Morrison (1967), and Sopher and Narramore (2000) are just some examples of compelling evidence that indifference curves systematically deviate from parallelness and straightness.

This paper is concerned with a nonparametric approach to the analysis of decisions on a probability simplex. The questions are (i) when individuals make decisions on subsets of a probability simplex, what testable conditions can be found to refute the hypothesis that individuals have quasiconcave preference on the probability simplex, and (ii) if the hypothesis is not refuted, how can we reconstruct bounds on the indifference curves? A key reference here is Machina (1985) who considers implications on choice behaviour when preferences are quasiconcave. The paper is also in the spirit of Varian's (1982, 1983) nonparametric approach to demand behaviour and the experimental approach of Andreoni and Miller (2002) and Choi et al. (2007b; see also Choi et al. 2007a and Fisman et al. 2007). A recent paper by Kalandrakis (2010) analyses choices of voters to test if voters' preferences can be represented by a concave utility function. Kalandrakis (2010). This paper is similar in the sense we try to find rationalising quasiconcave utility functions and consider ideal points; however Kalandrakis (2010) considers binary choices of voters, whereas we consider linear budgets. This paper also provides the tools for extensive experimental analysis of the preferences considered.

To refute quasiconcavity of preferences, it would be sufficient to find two lotteries between which an individual is indifferent, and then test if a linear combination of these lotteries is preferred to them. However, the method introduced in this paper allow the construction of simple yet powerful experiments to not only refute the hypothesis of quasiconcavity (without the need to find lotteries between which an individual is indifferent), but also to reconstruct bounds on the indifference curves, which can be used to further analyse the observed choices. For example, nonparametric interpersonal comparisons can be conducted, as is illustrated in Section 6 using data from Karni et al. (2008).

As we consider preferences on a probability simplex, the usual notion of monotonicity of preferences has to be dropped. It is replaced by the assumption that there is a single point of satiation (i.e., a unique maximiser) in the simplex. The paper sets out to derive testable implications and recoverability of preferences when this point is known. However, it is also shown that even if the point is not known, there are still testable implications; in particular, the strictly quasiconcave rationalisation theorem is unaffected. Furthermore, if the point is not known, we can test always the hypothesis that a certain point in the simplex is the unique maximiser. In that respect, the paper is also similar to Kalandrakis (2010), who provides a way to test voter ideal points.

The rest of the paper is organised as follows. Section 2 reviews two of the most relevant models for the framework considered here, specifically stochastic choice functions generated by deterministic preferences over lotteries and considerations for ex ante fairness. Section 3 introduces the notation and shows how to determine which part of the probability simplex is revealed worse to an observation. Section 4 uses the results of the previous section to show the analogy to the revealed preference approach for usual commodity spaces and competitive budgets. Three axioms are presented which closely resemble the Weak (Samuelson 1938), Strong (Houthakker 1950), and Generalised (Afriat 1967, Varian 1982) Axiom of Revealed Preference. The section gives constructive proofs in analogy to Afriat's Theorem to show that consistency with our Generalised (Strong) Axiom is equivalent to the existence of a (strictly) quasiconcave utility function which rationalises the observations. Section 5 shows how to reconstruct bounds on indifference curves through unobserved points. Section 6 illustrates the reconstruction of revealed preferred sets and the possibility of interpersonal comparisons with experimental data from Karni et al. (2008). Section 7 concludes. Appendix A contains the proofs.

2 Models

2.1 Stochastic Choice Generated by Deterministic Preferences over Lotteries

Stochastic choice has been studied by many researchers in the psychological and also in the economic literature. Early examples include Block and Marschak (1960) and Becker et al. (1963); Machina (1985) provides a list of references. More recently, stochastic choices have been analysed by Bandyopadhyay et al. (1999, 2002, 2004), Nandeibam (2009), and Heufer (2009, 2011b). The basic idea is that individuals have unstable or random preferences, or some important factors that influence choice are unobservable to the researcher and the choice behaviour therefore appears to have stochastic components. As Machina (1985) states,

> [t]he motivation for such an approach is clear: if when confronted with a choice over two objects the individual chooses each alternative a positive proportion of the time, it seems natural to suppose that this is because he or she 'prefers' each one to the other those same proportions of the time.

Machina, in the same paper, then goes on to provide "an alternative model of stochastic choice at the individual level". He assumes that individuals do not have stochastic preferences over pure outcomes but rather deterministic preferences over lotteries. If an individual chooses option A with probability p over option B, then he does not prefer A over B p proportion of time, but rather the individual actually prefers a lottery that yields A with probability p over any pure outcome.

Machina's interpretation of "stochastic choice" as deterministic preferences over lotteries has been experimentally tested against the aforementioned hypothesis by Sopher and Narramore (2000). They find support for Machina's idea; they report that

> [i]n general, subjects prefer mixtures of lotteries over extremes [...] Moreover, they are consistent over time, in the sense that the distribution of choices (for a given linear choice set) does not change very often [...] We interpret these results as supporting the deterministic preference version of stochastic choice over the random utility interpretation.

Albers et al. (2000) also find evidence that subjects have a preference for "chance" which cannot be explained by risk-averse or risk-seeking expected utility. Quasiconcave preference, i.e. preferences for randomization, have also been considered in Crawford (1990), Chew et al. (1991), Camerer (1992), Camerer and Ho (1994), and Starmer (2000), but the most detailed analysis of its implications can still be found in Machina (1985).

2.2 Individual Preferences for Ex Ante Fairness

Karni and Safra (2002a) (see also Karni and Safra 2002b) provide an axiomatic model of the behaviour of an individual with both self interest and preferences for fairness. The individual chooses a random allocation procedure; preferences for fairness imply convex indifference curves in a probability simplex, i.e. quasiconcave preferences. Imagine an experiment with three subject, one being a "dictator" who has to divide an indivisible good by assigning winning probabilities to each subject. A dictator with strong preferences for ex ante fairness might prefer $(p_1, p_2, p_3) = (1/3, 1/3, 1/3)$, whereas a selfish dictator might prefer $(p_1, p_2, p_3) = (1, 0, 0)$. Karni et al. (2008) investigate choice behaviour in such an experiment by offering subjects "budgets", i.e. line segments in the probability simplex.

These kind of preferences continue to received a lot of attention in the theoretical literature (see, e.g., Neilson 2006; Karni and Safra 2008; Sandbu 2008; Borah 2009) and experimental literature (e.g., Krawczyk and Le Lec 2008; Cappelen et al. 2010).

3 Setup: Budgets, Choices, and Revealed Preference

3.1 Preliminaries

This paper is concerned with decisions on hyperplanes as subsets of a set of lotteries when preferences are quasiconcave. First, we would like to find refutable conditions on observed choices which are hypothesised as generated by quasiconcave preferences. As will be seen later, we can not only find necessary and sufficient conditions for the existence of a quasiconcave utility function which rationalises the data, we can even construct such a utility function as in Afriat's (1967) theorem, similar to the constructions found in Varian (1982) and Matzkin and Richter (1991). Second, we would like to reconstruct boundaries on the indifference curves in the simplex which are implied by the observed choices.

As in Machina (1985), the *set of lotteries* over a set $A = \{a_1, \ldots, a_n\}$ of *distinct pure outcomes* is defined as

$$D(A) = \left\{ (p_1, \ldots, p_n) : p_i \in [0,1], \sum_{i=1}^{n} p_i = 1 \right\}, \tag{1}$$

where p_i is the probability that outcome a_i occurs. However, the following analysis will be easier when the basic ingredients are transformed into an equivalent framework which employs the so called Marschak-Machina triangle (or, more generally, simplex; cf. Marschak 1950, Machina 1982). First, define

$$\mathcal{X}_A = \left\{ x \in \mathbb{R}_+^{n-1} : \sum_{i=1}^{n-1} x_i \leq 1 \right\}. \tag{2}$$

Then some $x = (x_1, \ldots, x_{n-1}) \in \mathcal{X}_A$ corresponds to the lottery over A with probabilities $(x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i)$. As $D(A)$ and $\mathcal{X}_A$ are order isomorphic, all basic ingredients of the following analysis—budgets, choices, and preferences and utility functions— and the representation results in the next section apply to both $D(A)$ and $\mathcal{X}_A$. We will also drop the subscript A when the reference is clear. Let $int\mathcal{X}$ denote the interior of $\mathcal{X}$ and $\partial\mathcal{X} = \mathcal{X} \backslash int\mathcal{X}$ denote the boundary.

The primitives of the model are *observable choices* by some decision maker (DM) on subsets of $\mathcal{X}$. Thus, an observation is a set of lotteries over A which are available and the choice from this set made by the DM. We assume that generally the set of available alternatives consists of the convex hull of $n-1$ elements on the boundary of $\mathcal{X}$. That is, given a set $Q = \{q^1, \ldots, q^{n-1}\} \subset \partial\mathcal{X}$, the set of available alternatives is

$$B(Q) = \left\{ x \in \mathcal{X} : \exists \lambda \in [0,1]^{n-1}, \sum_{i=1}^{n-1} \lambda_i = 1, x = \sum_{i=1}^{n-1} \lambda_i q^i \right\}. \tag{3}$$

We will also refer to a set of available alternatives as a *budget*. Note that these budgets are hyperplanes in $D(A)$ which separate the simplex into two half-spaces.

The *choice function* C of a DM assigns to each budget $B(Q)$ a choice $C(Q) \in B(Q)$. These observations are the primitives of our analysis. Thus, a *set of observations* on a DM consists of m budgets $\{B(Q^i)\}_{i=1}^m$ and m choices $\{C(Q^i)\}_{i=1}^m$. These observations will be used to test certain hypothesis about the preferences of a DM.

The DM is assumed to have a utility function $V : \mathcal{X} \to \mathbb{R}$. The basic assumption is that the DM's choice over any subset of $\mathcal{X}$ corresponds to the lottery over the subset of A which maximises V, that is, $C(Q) = \{\arg\max_{x \in B(Q)} V(x)\}$.

A preference $\succsim \subset \mathcal{X} \times \mathcal{X}$ is a complete *pre-order*, that is, a binary relation which is complete, reflexive and transitive.[1] The symmetric part of $\succsim$ is denoted by $\sim$ and its asymmetric part is denoted by $\succ$, i.e. $x \sim y$ if $x \succsim y$ and $y \succsim x$,

[1] A pre-order is also called a quasi-order. The preference is complete if either $x \succsim y$ or $y \succsim x$ or both; it is reflexive if $x \succsim x$ for all x; it is transitive if $x \succsim y$ and $y \succsim z$ imply $x \succsim y$.

and $x \succ y$ if $x \succsim y$ and [not $y \succsim x$]. A preference is *quasiconcave* if for all $x, y \in \mathcal{X}$ $x \sim y$ implies $\lambda x + (1-\lambda) y \succsim y$ for $\lambda \in (0,1)$, or alternatively, if for all $y \in \mathcal{X}$ the set $\{x \in \mathcal{X} : x \succsim y\}$ is convex. It is *strictly quasiconcave* if $\{x \in int\mathcal{X} : x \succsim y\}$ is strictly convex.

We say that the function V *represents* the preference $\succsim$ if for all $x, y \in \mathcal{X}$, $x \succsim y$ implies $V(x) \geq V(y)$ and $x \succ y$ implies $V(x) > V(y)$. The first contribution of this paper is to provide necessary and sufficient conditions to test if there exists a utility function V which is maximised by the observed choices and represents a quasiconcave preference.

Quasiconcavity implies that the DM's indifference curves are convex. As the usual notion of monotonicity of preferences cannot be applied to preferences on a simplex, it needs to be replaced with a different meaningful assumption. Therefore, we assume that there is a unique $\succsim$-maximal element in $\mathcal{X}$, denoted by ω. The preferences we consider are therefore *single-peaked* in $\mathcal{X}$, and any utility function which represents a preference is *satiated* at ω and non-satiated at any $x \in \mathcal{X}$ such that $x \neq \omega$.[2]

The choice of a particular ω as the point of non-satiation is crucial for the rest of the paper, so it requires some discussion. The following analysis can be easily carried out whenever the point ω is known to the researcher. If it is unknown, then the analysis can be used to test possible choices of ω, that is, to test if a particular ω can indeed be the point of satiation.

The choice of ω depends on the context. If the elements of A are monetary outcomes, ω will be the degenerated lottery that assigns probability 1 to the maximal element in A (remember that we have assumed that the outcomes are distinct). If a point in the simplex represents an allocation of winning probabilities of a lottery for different individuals, a DM with strong preferences for ex ante fairness who gets to choose the point might prefer the element with equal probabilities, i.e. the centroid of the simplex. With weaker preferences for fairness, we would expect ω to be somewhere between the centroid and the degenerated lottery that assigns probability 1 to the winning probability of the DM. In these examples, ω can be deduced from normative considerations. But of course, ω ultimately depends on the DM's preferences. In an experimental setting, one could provide subjects one additional "budget" which consists of the entire set $\mathcal{X}$; their choice would then reveal ω. However, as will be shown in Section 4, it is also possible and perhaps more interesting to use the observed choices without knowledge about ω to test certain normative choices of ω. This will also be the focus of the application to experimental data in Section 6.

3.2 Auxiliary Budgets and the Revealed Preference Relation

The basic idea of *revealed preference* observed choices reveal something about a DM's preferences. Thus, the researcher constructs a *revealed preference relation* based on the observed choices. But before we can construct this relation, we need to take a closer look at the implications of the point ω.

[2]A utility function V is non-satiated at $x \in \mathcal{X}$ if there exists an $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ and $V(x) \geq V(y)$ for some $y \in \mathcal{X}$, where d is the Euclidean distance function; a utility function is satiated at ω if there does not exist an $\varepsilon > 0$ such that $d(\omega, y) > \varepsilon$ and $V(y) \geq V(\omega)$ for any $y \in \mathcal{X}$.

The budgets of the form $B(Q)$ separate the set of lotteries into two half spaces. Whether or not a choice on such a budget is preferred to some other point in $\mathcal{X}$ depends on ω; in particular, it depends on the half space in which ω is contained.

It is therefore helpful to describe budget with an implicit function. The sign of the implicit function, evaluated at some arbitrary point, can then be used to indicate whether or not ω is in the same half space as the point. We start by describing a budget with the following auxiliary definition:

$$\tilde{B} = \{x \in \mathcal{X} : \tilde{g}(x) = 0\} \tag{4}$$

with $\tilde{g} : \mathbb{R}_+^{n-1} \to \mathbb{R}$ being a continuous affine function, such that $\tilde{B} = B(Q)$. See Figure 1 for an example and Section A.1 in the appendix for a way to construct a function $\tilde{g}$ from a set Q.

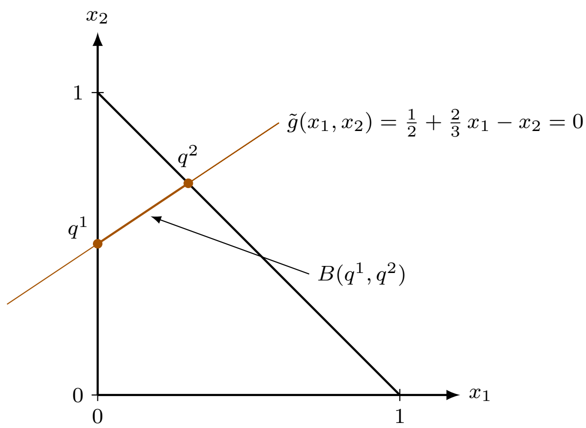

Figure 1: The Marschak-Machina triangle with $q^1 = (0, 1/2)$ and $q^2 = (3/10, 7/10)$. The point $(1, 0)$ corresponds to the lottery over A with $p_1 = 1$ and $p_2 = p_3 = 0$; the point $(0, 1)$ corresponds to $p_2 = 1$ and $p_1 = p_3 = 0$; the point $(0, 0)$ corresponds to $p_3 = 1$ and $p_1 = p_2 = 0$.

Because $\tilde{g}(x) = 0$ for all x on a budget, both $\tilde{g}(x)$ and $-\tilde{g}(x)$ describe the same budget. However, as described above, $\tilde{B}$ separates the simplex into two half spaces, and the two "sides" of $\tilde{B}$ can be interpreted economically: As ω is on one side of $\tilde{B}$ and if preferences are quasiconcave, then all elements on the other side of $\tilde{B}$ are considered worse than the optimal element on $\tilde{B}$.

We therefore define the function $g : \mathbb{R}^{n-1} \times \mathbb{R}^n \to \mathbb{R}$ as

$$g_\omega(x) = \begin{cases} \tilde{g}(x) & \text{if } \tilde{g}(\omega) > 0, \\ -\tilde{g}(x) & \text{if } \tilde{g}(\omega) < 0, \\ -|\tilde{g}(x)| & \text{if } \tilde{g}(\omega) = 0. \end{cases} \tag{5}$$

Then $g_\omega(x) < 0$ if x is on the "worse side" of the budget, and $g_\omega(x) > 0$ if x is not on the "worse side" of a budget. Furthermore, because we assume that ω is the unique maximal element, we need $g_\omega(x) < 0$ for all x which are not

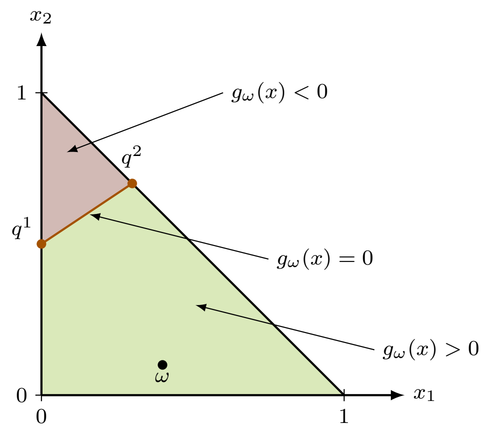

Figure 2: The Marschak-Machina triangle: Regions for $g_\omega(x) \lesseqgtr 0$.

on a budget if ω is an element of the budget, which is also guaranteed by the definition. See Figure 2 for an example.

Given a function $g^i_\omega(x)$ which describes the ith budget and depends on ω, auxiliary budgets $\bar{B}^i_\omega$ are then defined as

$$\bar{B}^i_\omega = \left\{x \in \mathcal{X} : g^i_\omega(x) = 0\right\}. \tag{6}$$

Both the definition of $\tilde{B}$ and $\bar{B}_\omega$ describe the same budget in the sense that the sets of alternatives available to the DM are the same. Thus, the auxiliary definition above describe the same budget, but it also provides information which is necessary to derive the revealed preference relation below.

We can now describe a set of m observations on a DM in terms of these auxiliary budgets, that is, as a set $\bar{S} = \{(\bar{B}^i_\omega, x^i)\}_{i=1}^m$, where $x^i \in \bar{B}^i_\omega$ is the choice of the DM on budget i, that is, $x^i = C(Q^i)$. Given the budgets, the choices, and ω, what elements are revealed worse than any x^i under the hypothesis of quasiconcave preferences? Define

$$\mathrm{dRW}_\omega(x^i) = \left\{x \in \mathcal{X} : g^i_\omega(x) \leq 0\right\} \tag{7}$$

$$\mathrm{sdRW}_\omega(x^i) = \left\{x \in \mathcal{X} : g^i_\omega(x) < 0\right\}. \tag{8}$$

We call these sets the set of elements which are *directly revealed worse* and *strictly directly revealed worse*, respectively. What motivates these definitions? Obviously, we cannot have $y \succ x^i$ when $y \in \bar{B}^i_\omega$ and x^i is the observed choice on $\bar{B}^i_\omega$. Consider Figure 3. Choose a lottery y in the set that does not contain ω, i.e. in the postulated set $\mathrm{sdRW}_\omega(x)$. We have $\omega \succ y$. Let $z = \lambda\,\omega + (1-\lambda)\,y$. Then by quasiconcavity, $z \succ y$. Now suppose $y \succsim x^i$. Then by transitivity, $z \succ x^i$. But $z \in \bar{B}^i_\omega$, that is, z was available to the DM when x^i was chosen. But then a DM with quasiconcave preferences would have chosen z instead of x^i, a contradiction.

The (indirectly) revealed worse set can be constructed in the following way: Suppose $x^j \in \mathrm{dRW}_\omega(x^i)$, so x^j is directly revealed worse than x^i. Then the set $\mathrm{dRW}_\omega(x^j)$ is indirectly revealed worse than x^i. Suppose $x^k \notin \mathrm{dRW}_\omega(x^i)$ but

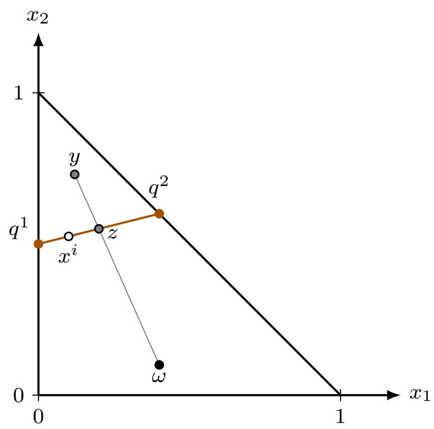

Figure 3: Illustration for the directly revealed worse sets.

$x^k \in \mathrm{dRW}_\omega(x^j)$. Then the set $\mathrm{dRW}_\omega(x_k)$ is also indirectly revealed worse than x^i. That is, the set $\mathrm{RW}_\omega(x^\ell)$ that is *revealed worse* than x^ℓ is the union of all $\mathrm{dRW}_\omega(x^i)$ for which either $x^i \in \mathrm{dRW}_\omega(x^\ell)$ or for some chain of observations with indices $i, j, k, \ldots, c$, we have $x^i \in \mathrm{dRW}_\omega(x^j)$, $x^j \in \mathrm{dRW}_\omega(x^k)$, ..., $x^c \in \mathrm{dRW}_\omega(x^\ell)$. The *strictly revealed worse* set $\mathrm{sRW}_\omega(x^i)$ can be defined analogously.

We can now define the *direct revealed preference relation* R_ω as

$$x^i \,\mathrm{R}_\omega\, x^j \quad \text{if} \quad g^i_\omega(x^j) \leq 0. \tag{9}$$

The (indirect) *revealed preference relation* R^*_ω is the transitive closure of R_ω, that is, the smallest transitive relation that contains R_ω. Furthermore, we define the *direct strictly revealed preference relation* P_ω as

$$x^i \,\mathrm{P}_\omega\, x^j \quad \text{if} \quad g^i_\omega(x^j) < 0. \tag{10}$$

The (indirect) *strictly revealed preference relation* P^*_ω is defined as

$$x^i \,\mathrm{P}^*_\omega\, x^j \quad \text{if for some indices } k \text{ and } \ell, x^i \,\mathrm{R}^*_\omega\, x^k \,\mathrm{P}_\omega\, x^\ell \,\mathrm{R}^*_\omega\, x^j. \tag{11}$$

4 Representation

4.1 Axioms: Refutable Conditions

Given our construction of the revealed preference relation R^*_ω, can we find refutable conditions for the hypothesis of quasiconcave preferences? Necessary conditions are easily found; however, as will be shown, we can also find conditions which are necessary and sufficient for the existence of a quasiconcave utility function that rationalises the observations.

Because of the assumption that ω is the unique maximal element, it is helpful to augment any set of observation by an observation ω—otherwise, we might have that for some observation i, $g^i_\omega(\omega) = 0$ and $x^i \neq \omega$, and as will be seen later, this is not a violation of the Generalised Axiom defined below. We will therefore augment any set of m observations by an observation consisting of a choice x^{m+1} on a budget $\bar{B}^{m+1}_\omega$ with g^{m+1}_ω defined as

$$g^{m+1}_\omega = -d(x, \omega), \tag{12}$$

where d is the Euclidean distance function, so $g^{m+1}_\omega(x) < 0$ for all $x \neq \omega$.

Let $M = \{1, \ldots, m+1\}$. We can now state our axioms.

Definition 1 *We say a set of observations* $\bar{S} = \{(\bar{B}^i_\omega, x^i)\}_{i \in M}$ *for distinct budgets* $\bar{B}^i_\omega$ *with constant* A *satisfies the* Weak Axiom of Revealed Quasiconcave Preference (Warq) *if for all* $\{i, j\} \subseteq M$ *such that* $x^i \neq x^j$

$$x^i \mathrm{R}_\omega x^j \quad \textit{implies} \quad g^j_\omega(x^i) > 0. \tag{13}$$

We say a set of observations satisfies the Strong Axiom of Revealed Quasiconcave Preference (Sarq) *if for all* $\{i, j\} \subseteq M$ *such that* $x^i \neq x^j$

$$x^i \mathrm{R}_\omega x^j \quad \textit{implies} \quad g^j_\omega(x^i) > 0. \tag{14}$$

It should be obvious that Warq implies Sarq but not vice versa.

Definition 2 *We say a set of observations satisfies the* Generalised Axiom of Revealed Quasiconcave Preference (Garq) *if for all* $\{i, j\} \subseteq M$

$$x^i \mathrm{R}_\omega x^j \quad \textit{implies} \quad g^j_\omega(x^i) \geq 0. \tag{15}$$

Clearly there exist data sets which violate Warq, Garq, or Sarq for some choices of ω. However, all three axioms—even Warq—are not empirically empty even when ω is arbitrarily chosen. The example in Figure 4 illustrates this.

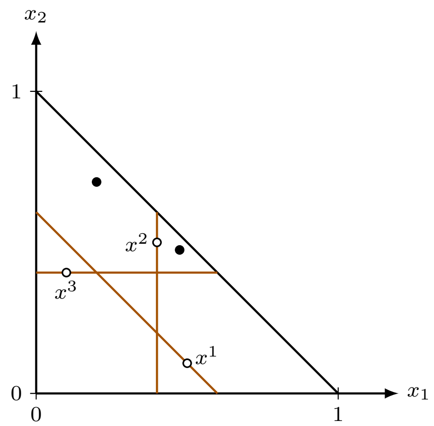

Figure 4: Three choices (circles); this set of observations violates Warq, Garq, and Sarqfor all ω. As an example, suppose ω is one of two dots. Then $x^1 \mathrm{R}_\omega x^3$ and $g^3_\omega(x^1) < 0$.

4.2 Representation

We will now proceed to show that our Generalised Axiom (Strong Axiom) is a necessary and sufficient condition for rationalisability or representation of the data by a continuous and (strictly) quasiconcave utility function. Because these two axioms are fairly easy to test in practice given a finite set of observations, they offer an efficient way to refute the hypothesis of (strictly) quasiconcave preferences.

Definition 3 *We say a function $U : \mathbb{R}^{n-1} \to \mathbb{R}$* rationalises *a set of observations $\bar{S} = \{(\bar{B}^i_\omega, x^i)\}_{i\in M}$ if*

$$U(x^i) \geq U(x) \quad \text{if} \quad x^i \, \mathrm{R}^*_\omega \, x \tag{16}$$

for all $i \in M$.

Note we can find a utility function on $\mathbb{R}^{n-1}$ which rationalises the data if and only if there exists a rationalising utility function on $D(A)$ because $D(A)$ and $\mathcal{X}$ are order isomorphic.

Theorem 1 *The following conditions are equivalent:*
(i) The set of observations $\bar{S} = \{(\bar{B}^i_\omega, x^i)\}_{i\in M}$ satisfies GARQ.
(ii) There exists a function $U : \mathcal{X} \to \mathbb{R}$ that is satiated at ω, non-satiated at every other element of $\mathcal{X}$, continuous and quasiconcave on int$\mathcal{X}$ and rationalises the set of observations $\bar{S}$.
(iii) There exist numbers $\{\phi_i, \lambda_i\}_{i\in M}$, $\lambda_i > 0$, such that for all $i, j \subseteq M$

$$\phi_j \leq \phi_i + \lambda_i \, g^i_\omega(x^j). \tag{17}$$

All proofs can be found in the appendix.

Theorem 2 *The following conditions are equivalent:*
(i) The set of observations $\bar{S} = \{(\bar{B}^i_\omega, x^i)\}_{i\in M}$ satisfies SARQ.
(ii) There exists a function $U : \mathcal{X} \to \mathbb{R}$ that is satiated at ω, non-satiated at every other element of $\mathcal{X}$, continuous and strictly quasiconcave on int$\mathcal{X}$ and rationalises the set of observations $\bar{S}$.
(iii) There exist numbers $\{\phi_i, \lambda_i\}_{i\in M}$, $\lambda_i > 0$, such that for all $i, j \subseteq M$

$$\phi_j < \phi_i + \lambda_i \, g^i_\omega(x^j) \qquad \text{for all } \{i,j\} \subseteq M \text{ with } x^i \neq x^j, \tag{18a}$$

$$\phi_j = \phi_i \qquad \text{for all } \{i,j\} \subseteq M \text{ with } x^i = x^j. \tag{18b}$$

4.3 What to do When the Most Preferred Lottery is Unknown

As discussed above, ω depends on the context. In many applications, such as lotteries over monetary payoffs, we can deduce ω from normative considerations. In other applications, the researcher may have no idea whatsoever about ω. For example, if the outcomes in A are modes of transportation and the DM likes variation, ω could be any point in $\mathcal{X}$.

If we do not know ω, GARQ and SARQ are still testable conditions: if an axiom is satisfied for *some* arbitrary $\omega \in \mathcal{X}$, we cannot reject the hypothesis of quasiconcavity of preferences. The problem is then to find an efficient way to test if there is such an ω. We can test sets $\{(\bar{B}^i_\omega, x^i)\}_{i\in M}$ for all $\omega \in \{x^i\}_{i=1}^m$. If this data set satisfies GARQ (SARQ), then obviously there exists a function $U : \mathcal{X} \to \mathbb{R}$ that is satiated at Ω^j, non-satiated at every other lottery, continuous and (strictly) quasiconcave on $\mathcal{X}$ and rationalises the set of observations.

For SARQ, we can go a step further and show that there exists a strictly quasiconcave rationalising utility function *if and only if* $S = \{(\bar{B}^i_\omega, x^i)\}_{i\in M}$ satisfies SARQ for some $\omega \in \{x^i\}_{i=1}^m$, as the following proposition shows.

Proposition 1 *The following conditions are equivalent:*

(i) The set of observations S satisfies SARQ *for some $\omega \in \mathcal{X}$.*

(ii) The set of observations S satisfies SARQ *with $\omega = x^i$ for some $x^i \in \{x^j\}_{j=1}^m$.*

5 Recoverability: Revealed Worse and Preferred Sets of Arbitrary Lotteries

It was shown in Section 3.2 how the revealed worse set can be constructed under the hypothesis of quasiconcave preferences. In this section, it is shown how one can construct the revealed worse and the revealed preferred set of arbitrary points in $\mathcal{X}$ which were not observed as choices. The analysis here closely follows Varian's (1982) approach.

Definition 4 *Given any point $x^0 \in \mathcal{X}$ not previously observed as a choice we define the set of budgets which* support x^0 *by*

$$\Theta_\omega(x^0) = \left\{\bar{B}^0_\omega : \{(\bar{B}^i_\omega, x^i)\}_{i\in M\cup\{0\}} \text{ satisfies GARQ and } g^0_\omega(x^0) = 0\right\}, \quad (19)$$

where

$$\bar{B}^0_\omega = \left\{x \in \mathcal{X} : g^0_\omega(x) = 0\right\}.$$

Note that Theorem 1 implies $\Theta_\omega(x^0)$ is non-empty for all x^0. Given $\Theta_\omega(x^0)$ we can easily describe the set of all points revealed worse than x^0: We require that for every x in the revealed worse set of x^0, we have that $x^0 \, \mathrm{P}^*_\omega \, x$ holds for all budgets in $\Theta_\omega(x^0)$ (e.g., if $x^0 \, \mathrm{P}^*_\omega \, x$ according to some $\bar{B}^0_\omega \in \Theta_\omega(x^0)$ but not according to some other $\bar{B}^0_\omega \in \Theta_\omega(x^0)$ then x is not in the revealed worse set). More succinctly, we define the revealed worse set of x^0, $\mathrm{RW}_\omega(x^0)$, as

$$\mathrm{RW}_\omega(x^0) = \left\{x \in \mathcal{X} : x^0 \, \mathrm{P}^*_\omega \, x \text{ for all } \bar{B}^0_\omega \in \Theta_\omega(x^0)\right\}. \quad (20)$$

Similarly, we can define the revealed preferred set of x^0, $\mathrm{RP}_\omega(x^0)$, by

$$\mathrm{RP}_\omega(x^0) = \left\{x \in \mathcal{X} : x \, \mathrm{P}^*_\omega \, x^0 \text{ for all } \bar{B}_\omega \in \Theta_\omega(x)\right\}. \quad (21)$$

Let $\mathrm{conv}(\{x^i\}_{i=1}^\ell)$ be the convex hull of a set of points. Define

$$\mathrm{CM}(x^0) = \mathit{int}\,\mathrm{conv}(\{x \in \{x^i\}_{i\in M} : x \, \mathrm{R}^*_\omega \, x^0\}), \quad (22)$$

and let $\overline{\mathrm{CM}}(x^0)$ be the closure of $\mathrm{CM}(x^0)$. Then the following can be shown to hold:

Proposition 2 $\mathrm{CM}(x^0) \subseteq \mathrm{RP}_\omega(x^0) \subseteq \overline{\mathrm{CM}}(x^0)$.

We omit the proof, as it is the same as in Varian (1982, Fact 12, p. 960) and Knoblauch (1992, Proposition 1, p. 661).

Because it is easy to check whether a point is in the convex hull of a set of points and to determine whether a point on the boundary of $\mathrm{RP}_\omega(x^0)$ belongs to $\mathrm{RP}_\omega(x^0)$, Proposition 5 completely describes $\mathrm{RP}_\omega(x^0)$. Furthermore, by definition $x^0 \in \mathrm{RW}_\omega(x)$ if and only if $x \in \mathrm{RP}_\omega(x^0)$, so we can easily determine whether or not a point x is in either $\mathrm{RW}_\omega(x^0)$ or $\mathrm{RP}_\omega(x^0)$.

6 Application

Karni et al. (2008) investigate choice behaviour in an experimental test of the models in Karni and Safra (2002a, 2002b; see also Section 2.2). Subjects were asked to choose a lottery on a budget (or chord, in their terminology) in a probability simplex which determined the probabilities with which each of three subjects would win a prize of a 15 USD. See Karni et al. (2008) for more details about the experimental setup. Note that not all of their subjects were given incentives; some subjects were asked to make merely hypothetical choices. Purely selfish preferences imply that the subject who chooses the lotteries always picks the lottery that gives him the highest probability of winning. Depending on their notion of fairness, subjects who prefer "fair" lotteries might pick, for example, the lottery that minimises the distance to the lottery which allocates equal probability to all three subjects, i.e. $(p_1, p_2, p_3) = (1/3, 1/3, 1/3)$ or $(x_1, x_2) = (1/3, 1/3)$. The linear self-interest preferences and the strictly quasiconcave fairness preferences combined yield quasiconcave preferences on the lotteries. Figure 5 shows the budgets used in the experiment.

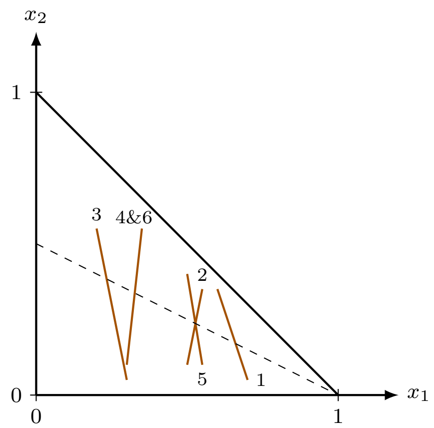

Figure 5: Budgets used by Karni et al. (2008).

Note that the budgets in the experiment do not start or end on the boundaries of the simplex. This is somewhat unfortunate, but does not lead to major problems. Choices in the relative interior of the budgets do not change anything, while some adjustments have to be made for choices on the endpoints. See Figure 6, which depicts the revealed worse set in case x is on the boundary of a budget.

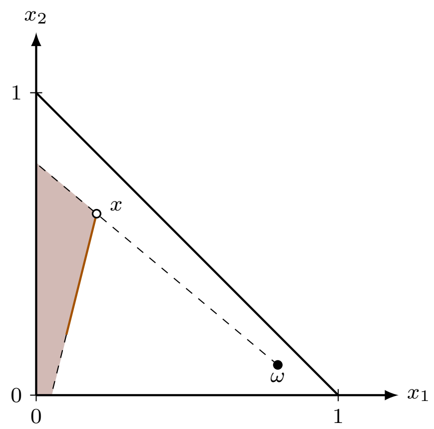

Figure 6: The revealed worse set for a choice on the boundary of the budget.

Furthermore, the experimental subjects were unfortunately not asked for their most preferred lottery on the entire simplex, but it is fairly reasonable to assume, based on normative considerations, that subjects were impartial towards the other two subject in the sense that they think of them as equally deserving. Hence, assuming that ω is a lottery with $p_2 = p_3$ appears sensible. Table 1 shows how many of the 135 subjects passed tests for GARQ and SARQ for different ω. Note that because budgets 4 and 6 were identical, GARQ and SARQ are not equivalent. However, without knowing the actual most preferred lottery, the test for GARQ with this experimental setup has no power whatsoever against the alternative hypothesis of purely random choice because only two budgets cross each other. Therefore the results in the table merely indicate how frequently the tested ω are consistent with the observed choices.

	ω				
	$(1, 0)$	$\left(\frac{3}{5}, \frac{1}{5}\right)$	$\left(\frac{2}{5}, \frac{3}{10}\right)$	$\left(\frac{11}{40}, \frac{29}{80}\right)$	$\left(\frac{3}{20}, \frac{17}{40}\right)$
Number of subjects satisfying GARQ	112	112	76	76	56
Number of subjects satisfying SARQ	70	70	45	45	34
	Number of subjects satisfying an axiom for at least one ω				
		GARQ		SARQ	
		135		87	

Table 1: Subjects consistent with GARQ and SARQ for different ω.

In Karni and Safra (2002a, 2002b) the authors decompose preferences $\succsim$ into a self-interest component $\succsim_S$ and a fairness component $\succsim_F$. They give the following definition: The pairs $(\succsim, \succsim_F)$ and $(\hat{\succsim}, \hat{\succsim}_F)$ are comparable if $\succsim_S = \hat{\succsim}_S$ and $\succsim_F = \hat{\succsim}_F$, i.e. if both incorporate the same idea of fairness and self-interest. For two comparable preference-fairness relation pairs $(\succsim, \succsim_F)$ and $(\hat{\succsim}, \hat{\succsim}_F)$ the pair $(\succsim, \succsim_F)$ possesses a *stronger sense of fairness* than $(\hat{\succsim}, \hat{\succsim}_F)$ if for every $x \in \mathcal{X}$, $P(x) \cap W_F(x) \subseteq \hat{P}(x) \cap \hat{W}_F(x)$, where $P(x) = \{y \in \mathcal{X} : y \succsim x\}$ and $W_F(x) = \{y \in \mathcal{X} : x \succsim_F y\}$.

Note that the revealed preferred sets used in this paper can be applied to the data collected by Karni et al. (2008) to make interpersonal comparisons. A detailed analysis is beyond the scope of the paper, but Figure 7 and 8 give two examples of interpersonal comparison.[3] Figure 7 shows the decision of subjects 10 and 71 and for illustration their respective set of lotteries revealed preferred to the lottery $(1/3, 1/3, 1/3)$, assuming that the most preferred point is the indicated ω. If we can accept that this lottery is the fairest one, then the set $W_F(1/3, 1/3)$ equals the entire simplex. The revealed preferred set of subject 71 is contained in the revealed preferred set of subject 10 (Figure 7); in fact, since all of subject 10's are closer to the boundaries of the budgets, this is the case for the revealed preferred sets to all lotteries in the simplex. Figure 8 shows the choices of subjects 71 and 93 where none of the two subjects possesses a stronger sense of fairness.

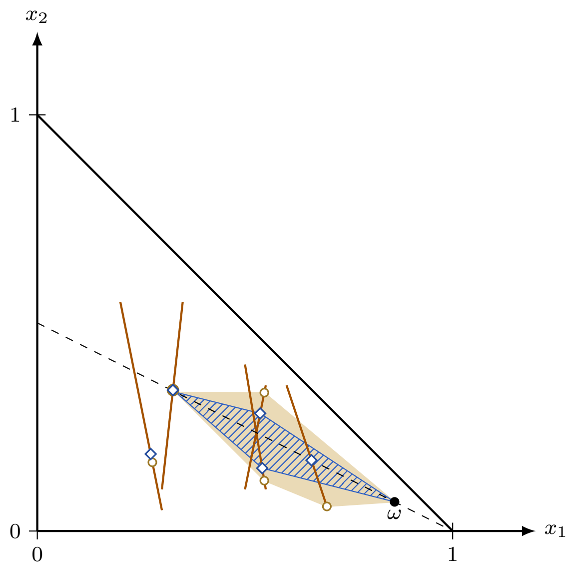

Figure 7: A stronger sense of fairness: Subject 10 (○, ▬) possesses a stronger sense of fairness than subject 71 (◇, ▨).

7 Discussion and Conclusions

This paper introduced a nonparametric approach to the analysis of decisions on a probability simplex. Easily testable necessary and sufficient conditions were found which guarantee the existence of a quasiconcave utility function which

[3]See also Heufer (2011a) for a similar comparative analysis of revealed risk preference. The data used here might also be useful for a revealed preference approach to comparative impartiality (Nguema 2003).

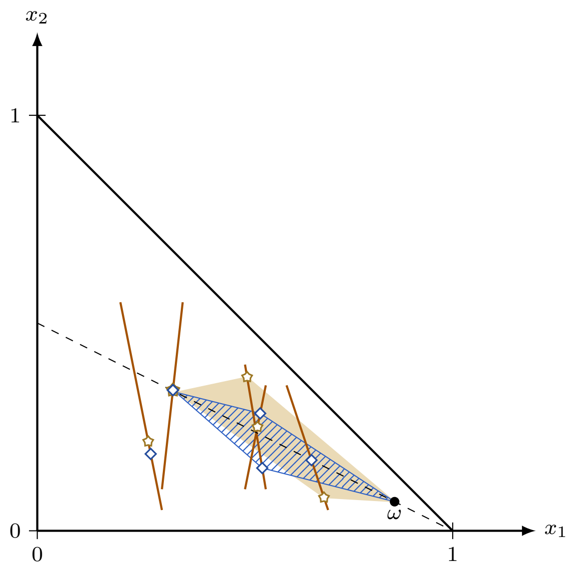

Figure 8: No stronger sense of fairness: Subject 93 (☆, ▬) does not possess a stronger sense of fairness than subject 71 (◇, ▨), and vice versa.

rationalises a set of observations. It was shown how one can construct an actual utility function, and how one can recover preferences. The analysis is much in the spirit of Afriat's (1967) and Varian's (1982) contribution to revealed preference and nonparametric demand analysis.

While the approach described here is in principle well suited for a laboratory experiment, there are practical issues which need to be addressed. First, unless the recruited subjects are students of fields likely to cover simplices, it would probably be impractical to attempt to explain subjects even what a probability simplex is. Second, the presentation of a hyperplane inside a tetrahedron would require subjects to choose at least two variables to determine a point on the hyperplane, and it is not clear in how far subjects would be aware of what they are doing. Also, a graphical presentation for lotteries over more than four outcomes might be difficult if not impossible.

As for the second point, an experimental investigation of choice behaviour should perhaps be restricted to three outcomes. The first point was already solved elegantly by Sopher and Narramore (2000) and Karni et al. (2008): Subjects were presented a slider on a computer screen which they used to determine the λ for the optimal combination of the two extreme lotteries q^1 and q^2, i.e. they could choose $\lambda\, q^1 + (1-\lambda)\, q^2$ with a simple mechanism. Their options were presented by a pie chart – a concept most subjects are probably familiar with.

A Appendix

A.1 Constructing the Implicit Budget Function

A function $\tilde{g}$ such that $\tilde{g}(x) = 0$ if and only if $x \in B(Q)$ can be easily found by solving a linear programming problem. Let α be a scalar and $\beta = (\beta_1, \ldots, \beta_{n-1})$. Remember that $Q = \{q^1, \ldots, q^{n-1}\}$ is a set of elements of $\mathcal{X}$, that is, $q^i = (q^i_1, \ldots, q^i_{n-1})$. Then solve

$$
\begin{aligned}
h^* = \min_{\alpha \in \mathbb{R}, \beta \in \mathbb{R}^{n-1}} \quad & 0 \cdot \alpha + 0 \cdot \beta \\
\text{subject to} \quad & \alpha + \sum_{j=1}^{n-1} (\beta_j \, q^i_j) = 0 \ \text{ for all } \ i = 1, \ldots, n-1 \\
& \alpha + \sum_{j=1}^{n-1} \beta_j = 1
\end{aligned}
\tag{23}
$$

and let $\tilde{g}(x) = \alpha + \sum_{j=1}^{n-1} (\beta_j x_j)$. Note that the constructed $\tilde{g}$ is not unique.

A.2 An Algorithm to Construct the Numbers in Theorem 1

This is a straightforward adaptation of Varian's (1982) algorithm to construct the Afriat numbers. We also need an algorithm which finds a maximal element of a binary relation T, called MaxElement(I, T), where $I = \{1, \ldots, m\}$ is a set of indices. An element x^μ of a set $\{x^i\}_{i \in I}$ is maximal with respect to a binary relation T if $x^i \, \mathrm{T} \, x^\mu$ implies $x^\mu \, \mathrm{T} \, x^i$. We can use Algorithm 2 in Varian (1982):

Algorithm 1
Input*: A reflexive and transitive binary relation* T *defined on a finite set* $\{x^i\}_{i \in I}$ *indexed by* $I = \{1, \ldots, m+1\}$.
Output*: An index* μ *where* $x^i \, \mathrm{T} \, x^\mu$ *implies* $x^\mu \, \mathrm{T} \, x^i$.

1. *Set* $\mu = 1$ *and* $y^0 = x^1$.
2. *For each* $i \in I$, *if* $x^i \, \mathrm{T} \, y^{i-1}$ *set* $y^i = x^i$ *and* $\mu = i$. *Otherwise set* $y^i = y^{i-1}$.

This algorithm correctly computes a maximal element (see Varian 1982, Fact 15).

Algorithm 2
Input*: A set of observations* $\{x^i\}_{i \in M}$ *and* $\{g^i_\omega(x)\}_{i \in M}$ *and the relation* R^*_ω *that satisfies* Garq.
Output*: A set of numbers* $\{\phi_i\}_{i \in M}$ *and* $\{\lambda_i\}_{i \in M}$.

1. *Set* $I = \{1, \ldots, m+1\}$, $J = \emptyset$.
2. *Let* $\mu =$ *MaxElement*(I, R^*_ω).
3. *Set* $E = \{i \in I : x^i \, \mathrm{R}^*_\omega \, x^\mu\}$. *If* $J = \emptyset$, *set* $\phi_\mu = \lambda_\mu = 1$ *and go to Step 6. Otherwise go to Step 4.*
4. *Set* $\phi_\mu = \min_{i \in E} \min_{j \in J} \min\{\phi_j + \lambda_j \, g^j_\omega(x^i), \phi_j\}$.
5. *Set* $\lambda_\mu = \max_{i \in E} \max_{j \in J} \max\{(\phi_j - \phi_\mu)/g^i_\omega(x^j), 1\}$.
6. *For all* $i \in E$, *set* $\phi_i = \phi_\mu$ *and* $\lambda_i = \lambda_\mu$.
7. *Set* $I = I \backslash E$, $J = J \cup E$. *If* $I = \emptyset$, *stop. Otherwise, go to Step 2.*

Lemma 1 *Algorithm 2 computes* $\{\phi_i\}_{i\in M}$ *and* $\{\lambda_i\}_{i\in M}$ *which satisfy the inequalities in Theorem 1, condition (iii).*

Proof Identical to the proof in Varian (1982), except that $p^i(x^j - x^i)$ is replaced by $g^i_\omega(x^j)$. ■

A.3 An Algorithm to Construct the Numbers in Theorem 2

Again, this is an adaptation of Varian's (1982) algorithm to construct the Afriat numbers, using additional ideas from Chiappori and Rochet (1987) and Matzkin and Richter (1991).

Algorithm 3
Input*: A set of observations* $\{x^i\}_{i\in M}$ *and* $\{g^i_\omega(x)\}_{i\in M}$ *and the relation* R^*_ω *that satisfies* SARQ.
Output*: A set of numbers* $\{\phi_i\}_{i\in M}$ *and* $\{\lambda_i\}_{i\in M}$.

1. *Set* $I = \{1, \ldots, m+1\}$, $J = \emptyset$, *and choose an* $\varepsilon > 0$.
2. *Let* $\mu = \mathit{MaxElement}(I, \mathrm{R}^*_\omega)$.
3. *Set* $E = \{i \in I : x^i \,\mathrm{R}^*_\omega\, x^\mu\}$. *If* $J = \emptyset$, *set* $\phi_\mu = \lambda_\mu = 1$ *and go to Step 6. Otherwise go to Step 4.*
4. *Set* $\phi_\mu = \min_{i\in E} \min_{j\in J} \min\{\phi_j + \lambda_j\, g_j(x^i) - \varepsilon, \phi_j - \varepsilon\}$.
5. *Set* $\lambda_\mu = \max_{i\in E} \max_{j\in J} \max\{(\phi_j - \phi_\mu + \varepsilon)/g^i_\omega(x^j), 1\}$.
6. *For all* $i \in E$, *set* $\phi_i = \phi_\mu$ *and* $\lambda_i = \lambda_\mu$.
7. *Set* $I = I \backslash E$, $J = J \cup E$. *If* $I = \emptyset$, *stop. Otherwise, go to Step 2.*

Lemma 2 *Algorithm 3 computes* $\{\phi_i\}_{i\in M}$ *and* $\{\lambda_i\}_{i\in M}$ *which satisfy the inequalities in Theorem 2, condition (iii).*

Proof We need to show the following:

(a) $\phi_i = \phi_j$ for all $j \in J$ and $i \in E$ such that $x^i = x^j$,

(b) $\phi_i = \phi_j$ for all $\{i, j\} \subseteq E$ such that $x^i = x^j$.

(c) $\phi_i < \phi_j + \lambda_j\, g^j_\omega(x^i)$ for all $j \in J$ and $i \in E$ such that $x^i \neq x^j$,

(d) $\phi_j < \phi_i + \lambda_i\, g^i_\omega(x^j)$ for all $j \in J$ and $i \in E$ such that $x^i \neq x^j$,

(e) $\phi_i < \phi_j + \lambda_j\, g^j_\omega(x^i)$ for all $\{i, j\} \subseteq E$ such that $x^i \neq x^j$,

At the first execution of the algorithm we have $J = \emptyset$. After Step 6 has been executed once, J contains only the "equivalent" indices in E, i.e. indices $i \in E$ such that $x^i \,\mathrm{R}^*_\omega\, x^\mu$. These elements are removed from I, such that at the second execution of Step 2, $\mu = \mathrm{MaxElement}(I, \mathrm{R}^*_\omega)$ cannot be in J. Indeed, after every execution of Step 6, μ at the next execution of Step 2 can never be in J.

Proof of (a): For all $i \in E$, we have $i \notin J$ because either $J = \emptyset$ by Step 1 or $J \cap I = \emptyset$ by Step 6. But if $\{i, j\} \subseteq I$ and $x^i = x^j$, then $\{i, j\} \subseteq E \subseteq I$, so $\{i, j\} \cap J = \emptyset$, hence the condition is always satisfied.

Proof of (b): If $x^i = x^j$, then either $\{i,j\} \subseteq E$ or $\{i,j\} \cap E = \emptyset$; furthermore, $\{i,j\} \cap J = \emptyset$. Then by Step 6 we have $\phi_i = \phi_j$.

Proof of (c): If $i \in E$, then $x^i \, \mathrm{R}^*_\omega \, x^\mu$. Because μ is a maximal element of I, $x^\mu \, \mathrm{R}^*_\omega \, x^i$. Since R^*_ω satisfies SARQ, we must have $x^i = x^\mu$ for all $i \in E$. At the first execution of Step 6 we have $\phi_i = \lambda_i = 1$ for all $i \in E$. After the first execution of Step 6, we can either use the proof for (a) respectively (b), or we have that $\{\mu\} = E$. In the latter case, we have by Step 4,

$$\phi_i \leq \phi_j + \lambda_j \, g^j_\omega(x^i) - \varepsilon$$

and with $\varepsilon > 0$ we have

$$\phi_i < \phi_j + \lambda_j \, g^j_\omega(x^i).$$

Proof of (d): Note that at Step 5 we must have $g^i_\omega(x^j) > 0$ for all $j \in J$. If that were not the case, $x^i \, \mathrm{R}^*_\omega \, x^j$ for some $j \in J$. But then i would have been moved to J before j was moved to J. Hence the division in Step 5 is well defined. We have

$$\lambda_i = \lambda_\mu \geq \frac{\phi_j - \phi_i + \varepsilon}{g^i_\omega(x^j)}.$$

Then

$$\lambda_i \, g^i_\omega(x^j) \geq \phi_j - \phi_i + \varepsilon$$

and with $\varepsilon > 0$ we have

$$\phi_j < \phi_i + \lambda^i \, g^i_\omega(x^j).$$

Proof of (e): If $\{i,j\} \subseteq E$, then $x^i \, R \, x^\mu$ and $x^\mu \, \mathrm{R}^*_\omega \, x^i$ because μ is a maximal element of I. Because R^*_ω satisfies SARQ, we must have $x^i = x^\mu$ for all $i \in E$, hence the condition is always satisfied. ■

A.4 Proofs

A.4.1 Proof of Theorem 1

Proof of Theorem 1: We proceed to show (ii) ⇒ (i), (i) ⇒ (iii), and finally (iii) ⇒ (ii) by construction of an actual utility function which rationalises the data. Even though the implicit functions describing the budgets are linear, they do replace the price vectors in the analysis of demand by competitive consumers. These price vectors are not present in our model. In that respect, the proof borrows from the generalisation of Afriat's Theorem due to Forges and Minelli (2009).

Proof of (ii) ⇒ (i): Varian (1982) shows that the existence of a rationalising non-satiated utility function implies GARP. For the non-satiated part of the utility function, the proof is the very similar (replacing $p^i x^i \geq p^i x^j$ with $g^i_\omega(x^j) < 0$ etc.; cf. Forges and Minelli 2009): Let $U(x)$ rationalise the data. If $x^i \,\mathrm{R}_\omega\, x^j$, then $U(x^i) \geq U(x^j)$; if $x^i \,\mathrm{R}^*_\omega\, x^j$, then there exist indices $(k, \ldots, \ell)$ such that $x^i \,\mathrm{R}_\omega\, x^k \,\mathrm{R}^*_\omega\, \ldots \,\mathrm{R}_\omega\, x^\ell \,\mathrm{R}_\omega\, x^j$, and $U(x^i) \geq U(x^k) \geq \ldots \geq U(x^j)$ implies $U(x^i) \geq U(x^j)$. We want to show that this implies $g^j_\omega(x^i) \geq 0$. Suppose first that $x^i \neq \omega$. If $g^j_\omega(x^i) < 0$, by the non-satiation of U we can find an $x \in \mathcal{X}$ such that $g^j_\omega(x) < 0$ and $U(x) > U(x^i) \geq U(x^j)$. But then U does not rationalise the data.

For the satiated part, suppose instead that $x^i = \omega$. Then $g^j_\omega(\omega) < 0$ is ruled out by the definition of g (Definition 5).

Proof of (i) ⇒ (iii): Lemma 1 shows that Algorithm 2 in Section A.2 computes the numbers.

Proof of (iii) ⇒ (ii): Define $V : \mathbb{R}^{n-1}_+ \to \mathbb{R}$ by

$$V(x) = \min_{i \in M} \left\{ \phi_i + \lambda_i \, g^i_\omega(x) \right\}. \tag{24}$$

Replacing the conditions involving price vectors in Varian (1982) with the functions g^i_ω shows that V is quasi-concave and rationalises the data for every $x \neq \omega$: As the minimum of finitely many concave and continuous functions, $V(x)$ is concave (and therefore quasiconcave) and continuous. To show that it rationalises the data, note that for all $j \in M$ we have $V(x^j) = \phi_j$. To see this, let $K = \left\{\arg\min_{i \in M} \{\phi_i + \lambda_i \, g^i_\omega(x^j)\}\right\}$. If $j \notin K$, then by (17) we have $\phi_j < \phi_k + \lambda_k \, g^k_\omega(x^j) = \min_{i \in M}\{\phi_i + \lambda_i \, g^i_\omega(x^j)\} = V(x^j)$. But since $V(x^j) = \min_{i \in M}\{\phi_i + \lambda_i \, g^i_\omega(x^j)\} \leq \phi_j + \lambda_j \, g^j_\omega(x^j) = \phi_j$, we have $\phi_j < V(x^j) \leq \phi_j$, a contradiction. For any x such that $g^j_\omega(x) \leq 0$ (i.e. $x^j \,\mathrm{R}_\omega\, x$) we have $V(x) \leq \phi_j + \lambda_j \, g^j_\omega(x) \leq \phi_j = V(x^j)$ and for any x such that $g^j_\omega(x) < 0$ (i.e. $x^j \,\mathrm{P}^*_\omega\, x$) we have $V(x) < \phi_j + \lambda_j \, g^j_\omega(x) \leq \phi_j = V(x^j)$. Finally, we have $V(\omega) = \phi_{m+1}$ because $\omega = x^{m+1}$, and for all $x \in \mathcal{X}$, $x \neq \omega$, we have $V(x) < \phi_{m+1}$. To see this, note that $\phi_{m+1} + \lambda_{m+1} \, g^{m+1}_\omega(x) < \phi_{m+1}$ by the definition of $g^{m+1}_\omega(x)$ in Eq. (12), so $\min_{i \in M}\{\phi_i + \lambda_i \, g^i_\omega(x)\} < \phi_{m+1}$. ∎

A.4.2 Proof of Theorem 2

Proof of Theorem 2: We proceed in the same way as in the proof to Theorem 1.

Proof of (ii) ⇒ (i): This part is not a straightforward adaptation as the proof in Matzkin and Richter (1991) for Sarp and strict concavity relies on their concept of "regular demand".

Let $U(x)$ rationalise the data. If $x^i \,\mathrm{R}_\omega\, x^j$, then $U(x^i) \geq U(x^j)$; if $x^i \,\mathrm{R}^*_\omega\, x^j$, then there exist indices $(k, \ldots, \ell)$ such that $x^i \,\mathrm{R}_\omega\, x^k \,\mathrm{R}_\omega\, \ldots \,\mathrm{R}_\omega\, x^\ell \,\mathrm{R}_\omega\, x^j$, and $U(x^i) \geq U(x^k) \geq \ldots \geq U(x^j)$ implies $U(x^i) \geq U(x^j)$. We want to show that this implies $g^j_\omega(x^i) > 0$. If $g^j_\omega(x^i) < 0$, by the non-satiation of U we can find an $x \in \mathcal{X}$ such that $g^j_\omega(x) < 0$ and $U(x) > U(x^i) \geq U(x^j)$. But then U does not rationalise the data. If $g^j_\omega(x^i) = 0$, then by strict quasiconcavity of U we have that for $y = \lambda \, x^i + [1 - \lambda] \, x^j$, $U(y) > \max\{U(x^i), U(x^j)\}$ so $U(z) > U(x^i)$. But $g^j_\omega(y) = 0$, which implies $U(x^j) \geq U(z)$, so $U(x^i) \geq U(x^j) \geq U(z) > U(x^i)$, a contradiction.

Proof of (i) ⇒ (iii): This can either be shown using a Theorem of the Alternative (Rockafellar 1970, Theorem 22.2, pp.198–199) in analogy to Matzkin and Richter (1991, Lemma 1) by replacing $\alpha_{ij} = p^i(x^j - x^i)$ in their paper with $g^i_\omega(x^j)$, or by means of a modification of the algorithm in Varian (1982, Algorithm 3). Lemma 2 shows that Algorithm 3 in Section A.3 computes the numbers.

Proof of (iii) ⇒ (ii): We follow Matzkin and Richter (1991) in constructing the utility function. Let $T > 0$ and define $f : \mathbb{R}^{n-1}_+ \to \mathbb{R}$ by

$$f(x_1, \ldots, x_{n-1}) = \left[\sum_{i=1}^{n-1} (x_1)^2 + T\right]^{\frac{1}{2}} - T^{\frac{1}{2}}. \tag{25}$$

There exists an $\varepsilon_0 > 0$ such that $\phi_j < \phi_i + \lambda_i \, g^i_\omega(x^j) - \varepsilon_0$ for all $\{i, j\} \subseteq M$ with $x^i \neq x^j$ and the other two conditions of Theorem 2 (iii) hold as well, as can also be seen in the proof of Lemma 2. Then we can choose an ε so small that

$$\phi_j < \phi_i + \lambda_i \, g^i_\omega(x^j) - \varepsilon \, f(x^j - x^i) \qquad \text{for all } \{i, j\} \subseteq M \text{ with } x^i \neq x^j, \tag{26a}$$

$$\lambda_i > 0 \qquad \text{for all } i \in M, \tag{26b}$$

$$\phi_i = \phi_j \qquad \text{for all } \{i, j\} \subseteq M \text{ with } x^i = x^j. \tag{26c}$$

For each $i \in M$ we define $\pi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\pi_i(x) = \phi_i + \lambda_i \, g^i_\omega(x) - \varepsilon \, f(x - x^i). \tag{27}$$

Clearly, f is strictly convex, so each π_i is strictly concave. Furthermore, $\pi_i(x^i) = \phi_i$, because $f(x) = 0 \Leftrightarrow x = \mathbf{0}$. Now define $V : \mathbb{R}^{n-1}_+ \to \mathbb{R}$ by

$$V(x) = \min_{i \in M} \{\pi_i(x)\}. \tag{28}$$

As the minimum of finitely many strictly concave and continuous functions, $V(x)$ is strictly concave (and therefore strictly quasiconcave) and continuous.

To show that it rationalises the data, note that for all $j \in M$ we have $V(x^j) = \phi_j$. To see this, let $K = \left\{\arg\min_{i \in M}\{\pi_i(x^j)\}\right\}$. If $j \notin K$, then by (18a) we have $\phi_j < \pi_k(x^j) = \min_{i \in M}\{\pi_(x^j)\} = V(x^j)$. But since $V(x^j) = \min_{i \in M}\{\pi_i(x^j)\} \leq \pi_j(x^j) = \phi_j$, we have $\phi_j < V(x^j) \leq \phi_j$, a contradiction. For any x such that $g^j_\omega(x) \leq 0$ (i.e. $x^j \, \mathrm{R}_\omega \, x$) we have $V(x) < \pi_j(x) \leq \phi_j = V(x^j)$ and obviously for any x such that $g^j_\omega(x) < 0$ (i.e. $x^j P_A x$) we have $V(x) < \pi_j(x) \leq \phi_j = V(x^j)$. Finally, we have $V(\omega) = \phi_{m+1}$ because $\omega = x^{m+1}$, and for all $x \in \mathcal{X}$, $x \neq \omega$, we have $V(x) < \phi_{m+1}$. To see this, note that $\pi_{m+1}(x) < \phi_{m+1}$ by the definition of $g^{m+1}_\omega(x)$ in Eq. (12), so $\min_{i \in M}\{\pi_i(x)\} < \phi_{m+1}$. ■

A.4.3 Proof of Proposition 1

Proof of Proposition 1: (ii) ⇒ (i) is obvious. We will prove (i) ⇒ (ii).

Let $x^i \in \{x^k\}^m_{k=1}$ be some choice such that no other choice is strictly preferred to it; such an x^i exists if the data satisfy SARQ for some ω. Then, as there exists a strictly quasiconcave utility U function which rationalises the data, $U(\omega) > U(x^i)$, but also $U(\omega) > U(x)$ for all $x = \lambda x^i + (1-\lambda)\omega$ with $\lambda \in (0,1]$. Thus, $x^j \mathrm{R}^*_\omega x$ for all $x = \lambda x^i + (1-\lambda)\omega$ with $\lambda \in (0,1)$ is impossible for all $x^j \in \{x^k\}_{k=1}^m$. But then we can let $\omega = x^i$. ■

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