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Testing Revealed Preferences for Homotheticity with Two-Good Experiments

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Abstract It is shown that for two dimensional commodity spaces any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set. The result is used to provide a simplified nonparametric test for homotheticity of demand and a measure for homothetic efficiency. The article thus provides a useful tool to screen data for severe violations of homotheticity before estimating parameters of homothetic utility functions. The new test and measure are applied to previously published data.

Journal of Economic Literature Classifications: C14; C91; D11; D12.

Keywords Homothetic efficiency; Homothetic preferences; Laboratory experiments; Nonparametric tests; Revealed preference

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1 Introduction

Homotheticity of consumer preferences features importantly in theory and applications. If preferences are homothetic, a consumer's entire preference relation can be deduced from a single indifference set. Assuming homothetic preferences provides useful restrictions for the analysis of consumer demand.[1]

In applications researchers often focus on special types of homothetic preferences, like those given by a CES utility function. A researcher who wishes to estimate homothetic demand functions using consumption data might wish to test if the data could have been generated by a homothetic utility function.

A common nonparametric test of the utility maximization hypothesis has been developed by Afriat (1967) and refined by Varian (1982; 1983). In applications, especially in laboratory experiments, the commodity space is often only two dimensional.[2] For the two-commodity case Rose (1958) showed that satisfying the Weak Axiom of Revealed Preference (WARP) as introduced by Samuelson (1938) is sufficient for utility maximization. Banerjee and Murphy (2006) used the result to develop a simplified test for utility maximization.

In this article it is first shown that Rose's result carries over to homothetic rationalization: In the two-commodity case pairwise testing of observations is sufficient to test for consistency with a homothetic utility function. The result is stated as a pairwise version of Varian's (1983) Homothetic Axiom of Revealed Preference (HARP); testing this new sufficient axiom is much faster, which is useful for extensive Monte-Carlo simulations. The result also provides a direct way to compute scalar factors needed to construct the set of all bundles homothetically revealed preferred to any bundle.

Knowing that pairwise comparison is sufficient also allows for the definition of a homothetic efficiency index, which can be usefully applied to screen data for severe violations of homotheticity. The test and measure for homothetic efficiency developed in this article are applied to data sets from two-person dictator experiments by Andreoni and Miller (2002) and Fisman et al. (2007) and to a two-asset risk preferences experiment by Choi et al. (2007). In the first two articles the authors use the collected data to estimate parameters of a CES utility function, so the question of the validity of their (implicitly maintained) hypothesis is quite sensible in this context.

The remainder is organized as follows. Section 2 reviews the relevant part of revealed preference theory and introduces the Pairwise Homothetic Axiom of Revealed Preference (PHARP) and shows that in two dimensions PHARP is equivalent to HARP. Section 3 provides a simplified test for homotheticity

[1] Homothetic preferences are also important when dealing with aggregation problems (e.g. Chipman 1974). Characterisation and conditions for existence of homothetic functions have been considered (e.g. Dow and da Costa Werlang 1992, Candeal and Induráin 1995, Bosi 1998). The importance of homotheticity in economics has also generated a literature on testing data for homotheticity (e.g. Hanoch and Rothschild 1972, Varian 1983, Silva and Stefanou 1996, Liu and Wong 2000).

[2] See, for example, Harbaugh and Krause (2000), Harbaugh et al. (2001), Andreoni and Miller (2002), Chen et al. (2006), Choi et al. (2007), Fisman et al. (2007), Banerjee and Murphy (2007), Dickinson (2009), Dawes et al. (2011).

and a way to measure the extent of deviation from homotheticity. In Section 4 these ideas are applied to previously published data sets. Section 5 concludes. The supplementary material contains some illustrations and more details on the data analysis.

2 Theory

2.1 Preliminaries

Let $\mathbb{R}^\ell_+$ be the commodity space, where $\ell \geq 2$ denotes the number of different commodities.[3] The price space is $\mathbb{R}^\ell_{++}$, and the space of price-income vectors is $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$. Budget sets are of the form $B(\mathbf{p}, w) = \{\mathbf{z} \in \mathbb{R}^\ell_+ : \mathbf{p}\,\mathbf{z} \leq w\}$ for some price vector $\mathbf{p} \in \mathbb{R}^\ell_{++}$ and income $w > 0$. The demand function $D(\mathbf{p}, w)$ of a consumer assigns to each budget set the commodity bundle chosen by the consumer. Demand is exhaustive, i.e. $\mathbf{p}\,\mathbf{z} = w$. We assume to have a finite set of $n \geq 2$ observations; an observation is a triple $(\mathbf{z}, \mathbf{p}, w)$ where $\mathbf{z} = D(\mathbf{p}, w)$. The set of all observations on a consumer is then $\{(\mathbf{z}^i, \mathbf{p}^i, w^i)\}_{i=1}^n$.

A bundle $\mathbf{z}^i$ is *directly revealed preferred* to $\mathbf{z}$, written $\mathbf{z}^i R^0 \mathbf{z}$, if $w^i \geq \mathbf{p}^i \mathbf{z}$. It is *revealed preferred* to $\mathbf{z}$, written $\mathbf{z}^i R\,\mathbf{z}$, if for some sequence of bundles it is the case that $\mathbf{z}^i R^0 \mathbf{z}^j$, $\mathbf{z}^j R^0 \mathbf{z}^k$, …, $\mathbf{z}^m R^0 \mathbf{z}$. A bundle $\mathbf{z}^i$ is *strictly directly revealed preferred* to a bundle $\mathbf{z}$, written $\mathbf{z}^i P^0 \mathbf{z}$, if $w^i > \mathbf{p}^i \mathbf{z}$. A utility function $u(x)$ *rationalizes* a set of observations if $u(\mathbf{z}^i) \geq u(\mathbf{z})$ for all $\mathbf{z}$ such that $\mathbf{z}^i R^0 \mathbf{z}$ for all $i = 1, \ldots, n$.

The data satisfy the Weak Axiom of Revealed Preference (WARP) if for all pairs of observations i, j, $\mathbf{z}^i \neq \mathbf{z}^j$, $\mathbf{z}^i R^0 \mathbf{z}^j$ implies $\left[\text{not } \mathbf{z}^j R^0 \mathbf{z}^i\right]$. The data satisfy the Strong Axiom of Revealed Preference (SARP) if for all pairs of observations i, j, $\mathbf{z}^i \neq \mathbf{z}^j$, $\mathbf{z}^i R\,\mathbf{z}^j$ implies $\left[\text{not } \mathbf{z}^j R^0 \mathbf{z}^i\right]$. The data satisfy the Generalized Axiom of Revealed Preference (GARP) if for all pairs of observations i, j, $\mathbf{z}^i R\,\mathbf{z}^j$ implies $\left[\text{not } \mathbf{z}^j P^0 \mathbf{z}^i\right]$. Let $\mathcal{U}$ be the set of all continuous, monotonic, and concave utility functions. GARP (SARP) is a necessary and sufficient condition for the existence of a (strictly concave) $u \in \mathcal{U}$ that rationalizes the data (see Afriat 1967, Varian 1982, Matzkin and Richter 1991).

A utility function is *homothetic* if it is a positive monotonic transformation of a utility function that is homogeneous of degree 1. The data satisfy the Homothetic Axiom of Revealed Preference (HARP) if for all distinct choices of indices $(i, \ldots, m)$, $(\mathbf{p}^i \mathbf{z}^j)(\mathbf{p}^j \mathbf{z}^k) \cdots (\mathbf{p}^m \mathbf{z}^i) \geq w^i w^j w^k \cdots w^m$. HARP is a necessary and sufficient condition for the existence of a homothetic $u \in \mathcal{U}$ which rationalizes the data (see Varian 1983).

The data satisfy the Weak Generalized Axiom of Revealed Preference (WGARP) if for all pairs of observations i, j, $\mathbf{z}^i R^0 \mathbf{z}^j$ implies $\left[\text{not } \mathbf{z}^j P^0 \mathbf{z}^i\right]$. For two dimensions, Rose (1958) showed that WARP and SARP are equivalent, and Banerjee and Murphy (2006) showed that WGARP and GARP are equivalent.

[3] Notation: $\mathbb{R}^\ell_+ = \{\mathbf{z} \in \mathbb{R}^\ell : \mathbf{z} \geq 0\}$, $\mathbb{R}^\ell_{++} = \{\mathbf{z} \in \mathbb{R}^\ell : \mathbf{z} \gg 0\}$, where " $\mathbf{z} \geq 0$ " means " $z_i \geq 0$ for all i ", and " $\mathbf{z} \gg 0$ " means " $z_i > 0$ for all i ". Subscripts are used to denote scalars or vector components and superscripts to index bundles.

2.2 Homotheticity and Two-Commodity Choice

Following Knoblauch (1993), assume that a set of observations satisfy HARP, and define a scalar $t^{i,j}$ for all i, j, as

$$t^{i,j} = \min\left\{[(\mathbf{p}^i\mathbf{z}^k)/w^i]\,[(\mathbf{p}^k\mathbf{z}^\ell)/w^k]\cdots[(\mathbf{p}^m\mathbf{z}^j)/w^m]\right\}, \qquad (1)$$

where the minimum is over all distinct choices of indices $k, l, \ldots, m$, and $t^{i,i} = 1$. Then $t\,\mathbf{z}^i$ is *homothetically revealed preferred* to $\mathbf{z}^j$, written $t\,\mathbf{z}^i\,H\,\mathbf{z}^j$, if $t \geq t^{i,j}$. Note that $t = t^{i,j}$ is the smallest scalar for which $t\,\mathbf{z}^i\,H\,\mathbf{z}^j$.

When the consumption space is two dimensional, the budgets can be ranked by the price ratio. Let $\mathbf{z}^i = (x^i, y^i)'$ and choose good x as the numeraire. Then $\mathbf{p}^i = (1, q^i)$, where q^i is the relative price of good y. Let the income w^i be redefined appropriately. Without loss of generality, let the data be ordered by q such that $q^i \geq q^{i+1}$. If there are observations with the same q, let them be ordered such that $y^i/x^i \leq y^{i+1}/x^{i+1}$.

Homotheticity implies that income expansion paths are straight lines through the origin. It is easy to show that the slope of the expansion path, y/x, must increase as the relative price of y decreases: In the case of homotheticity, $(\mathbf{p}^i\mathbf{z}^j)(\mathbf{p}^j\mathbf{z}^i) \geq (\mathbf{p}^i\mathbf{z}^i)(\mathbf{p}^j\mathbf{z}^j)$. That is equivalent to $(q^i - q^j)(x^iy^j - y^ix^j) \geq 0$. If $i < j$, then $(q^i - q^j) \geq 0$, so it must be that $(x^iy^j - y^ix^j) \geq 0$. Thus $y^i/x^i \leq y^j/x^j$, and analogously for $i > j$. This is obviously a necessary condition for homotheticity, but it is not obvious that it is also sufficient.

We can now introduce a new axiom for homothetic choice, the PHARP, and show that in two dimensions, PHARP is already sufficient for the existence of a homothetic utility function that rationalizes the data: The data satisfy the Pairwise Homothetic Axiom of Revealed Preference (PHARP) if for all distinct choices of indices i, j, $(\mathbf{p}^i\mathbf{z}^j)(\mathbf{p}^j\mathbf{z}^i) \geq w^iw^j$. We can now present the first result; the proof can be found in the appendix.

Lemma 1 *For a two-dimensional commodity space and a set of data which satisfies PHARP and is ordered by q such that $q^i \geq q^{i+1}$, define*

$$\theta^{i,j} = \prod_{k=j}^{i-1} \frac{\mathbf{p}^{k+1}\mathbf{z}^k}{\mathbf{p}^{k+1}\mathbf{z}^{k+1}} \text{ if } i > j \quad \text{and} \quad \theta^{i,j} = \prod_{k=i}^{j-1} \frac{\mathbf{p}^k\mathbf{z}^{k+1}}{\mathbf{p}^k\mathbf{z}^k} \text{ if } i < j.$$

Then $\theta^{i,j} = t^{i,j}$.

Theorem 1 *For two-dimensional commodity spaces the following conditions are equivalent: (1) there exists a homothetic $u \in \mathcal{U}$ that rationalizes the data; (2) the data satisfy HARP; (3) the data satisfy PHARP.*

2.3 Rationalizability Conditions and Experimental Data

Suppose we observe several choices on different budgets of subjects in a laboratory experiment. If a set of observations on a consumer is found to be consistent with GARP (HARP), the conclusion that there exists a (homothetic) utility function which rationalizes a subjects' choices relies on the assumption that all choice variables are observed. With a proper subsets of goods, the utility maximization hypothesis imposes no restrictions on the data (see Varian 1988), unless we additionally assume that utility is weakly separable. Thus, if we let $\mathbf{o}$ denote unobserved consumption outside the laboratory and $\mathbf{z} \in \mathbb{R}^2_+$ the choice in a laboratory experiment with a two-dimensional commodity space, then WGARP (PHARP) applied to the experimental data is a necessary and sufficient condition for the existence of a rationalizing (homothetic) "subutility function" $u(\mathbf{z})$, but only a necessary condition for the existence of a rationalizing (homothetic) "macro utility function" $v(\mathbf{o}, u(\mathbf{z}))$ (see Varian 1983, 1988).

Observing only a subset of goods can arise if consumers' savings decisions or labour supply is not observed (Cox 1997). This problems are of little relevance in the laboratory experiments considered in Section 4 as they were designed to mimic a closed economy: The endowment of subjects was exogenous and could not be saved or spent on any goods not specified by the experimenter. But wealth, decisions, and other circumstances outside the laboratory can influence the decisions in the experiment.

Observing two-dimensional data will, realistically, always only contain a subset of goods. The methods in Section 3 and the results in Section 4 should therefore be interpreted in terms of subutility rationalization.

3 Applications

3.1 Testing HARP and PHARP

Testing HARP requires shortest path algorithms such as the Floyd-Warshall algorithm (Warshall 1962, Floyd 1962), which can detect negative weight cycles in the data: Varian (1983) interprets the set of n observations as a weighted graph with n nodes and an associated n^2 cost matrix $C = \{c_{ij}\}$, where $c_{ij} = \log \mathbf{p}^i \mathbf{z}^j / w^i$ is interpreted as the 'cost' of moving from node i to node j. HARP then requires that moving from a node i to itself is never 'cheaper' than zero (i.e., there are no negative weight cycles).

The Floyd-Warshall algorithm takes $\{c_{ij}\}$ as input and, if there are no negative weight cycles, outputs a matrix $\{\bar{c}_{ij}\}$ which is the minimum cost of moving from node i to node j; otherwise, it will detect that there is a cycle. Note that $\exp(\bar{c}_{ij}) = t^{i,j}$ if the data satisfy HARP. The complexity of the algorithm is $\Theta(n^3)$. In the case of two dimensions, we know from Theorem 2 that pairwise comparison is sufficient to detect violations of homotheticity. Hence a faster way to test if a set of consumption data satisfies homotheticity is to compute the matrix $\mathbf{M} = \{m_{ij}\}$, where $m_{ij} = (\mathbf{p}^i \mathbf{z}^j)(\mathbf{p}^j \mathbf{z}^i)/(w^i\, w^j)$; the PHARP is violated if and only if any element of $\mathbf{M}$ is less than 1. This test is very simple to compute, and the complexity is $\Theta(n^2)$, which for many observations can save a lot of time when doing Monte-Carlo simulations to approximate the power of the test against random behavior.

3.2 Homothetic Efficiency

If the data does not satisfy an axiom, one might like to know of how severe this violation is. The GARP can still be usefully applied using, for example, the Afriat Efficiency Index (AEI, see Afriat 1972), which is a popular measure and often reported in experimental studies: Define a relation $R^0(e)$ as $\mathbf{z}^i R^0(e)\mathbf{z}^j$ if $e\,w^i \geq \mathbf{p}^i\mathbf{z}^j$, and let $R(e)$ be the transitive closure of $R^0(e)$. Then the AEI is the greatest $e \in [0,1]$ for which $\mathbf{z}^i R(e)\mathbf{z}^j$ implies [not $e\,w^j > \mathbf{p}^j\mathbf{z}^i$].

A similar efficiency index for homotheticity would have to be based on the scalar factors t for which $t\,\mathbf{z}^i\,H^0\,\mathbf{z}^j$. But violation of HARP implies that there are negative weight cycles in the matrix C, so the 'cost' of moving from i to itself can be made arbitrarily cheap, and applying the Floyd-Warshall algorithm will not give any usable results. For two dimensions, however, we can exploit the fact that pairwise comparison is sufficient: An $m_{ij} < 1$ in the $\mathbf{M}$ matrix defined above can be interpreted as a measure for wasted income. To see this, note that homotheticity implies that the demand at $B(\mathbf{p}^i, \mathbf{p}^i\mathbf{z}^j)$ must be $[(\mathbf{p}^i\mathbf{z}^j)/w^i]\,\mathbf{z}^i$. Then if PHARP is violated with $m_{ij} < 1$, we have that $w^j > \mathbf{p}^j\,[(\mathbf{p}^i\mathbf{z}^j)/w^i]\,\mathbf{z}^i$, i.e. $\mathbf{z}^j$ is strictly revealed preferred to a bundle which is homothetically revealed preferred to $\mathbf{z}^j$.

Rearranging gives $1 > [(\mathbf{p}^j\mathbf{z}^i)/w^j][(\mathbf{p}^i\mathbf{z}^j)/w^i]$, which is precisely what the test in Section 3.1 finds if PHARP is violated. Put differently, a homothetic consumer could have obtained the same utility as he obtained from $\mathbf{z}^j$ at an expenditure below w^j; this expenditure is given by $m_{ij}\,w^j$. Thus, if PHARP is violated, it seems natural to use the lowest $m_{ij} \in \mathbf{M}$ as an efficiency index. More formally, we define the homothetic efficiency index as

$$\mathrm{HE} = \min\big\{1, \min_{i,j\in\{1,\ldots,n\}}\{m_{ij}\}\big\}. \tag{2}$$

A figure illustrating the idea can be found in the supplemenary material. It can also be helpful to look at the observation-wise partial index to see if it is only a particular observation which is responsible for a low value of HE. Define

$$\tilde{\mathrm{HE}}(i) = \min\big\{1, \min_{j\in\{1,\ldots,n\}}\{m_{ij}\}\big\}. \tag{3}$$

There is no 'natural' level of the HE which can be interpreted as 'close enough' to homotheticity. But by drawing many random choices as proposed by Bronars (1987) and computing the HE for each simulated set of observations, we obtain an approximation of the distribution of the HE under random choice, which can then be compared to the HE of real observations. This is done in Section 4.

4 Applications to Experimental Data

The experiment of Andreoni and Miller (2002) (AM) was designed to test the (subutility) rationalisability of altruistic choices. It is a generalized dictator game in which one subject (the dictator) allocates token endowments between himself and an anonymous other subject with different transfer rates. The payoffs of the dictator and the beneficiary are interpreted as two distinct goods, and the transfer rates as the price ratio. Fisman et al. (2007) (FKM) employed the same idea as AM, but presented budgets graphically. Subjects could then make a choice by clicking on their desired bundle with a computer mouse, which allowed to collect a large set of observations per subject. Note that in both articles the authors do not only test for (subutility) rationalisability but also use the collected data to estimate the parameters of a CES utility function, which is homothetic. Thus they implicitly maintain the hypothesis that their subject have homothetic preferences, or at least are "close enough" to homotheticity. Choi et al. (2007) (CFGK) used graphical presentations of portfolio choice problems to study behaviour under uncertainty at the individual level. The two goods in this case were two assets which paid off only with a certain probability.

As mentioned in Section 3.2, there is no natural critical value of the HE. We therefore use a Monte Carlo approach and simulate choice data in the following way: For each of the budgets presented to each of the subjects, draw random choices from a uniform distribution on the budgets. Then, for each of these data sets, compute the HE. This approximates the distribution of the HE under the hypothesis of random behavior. We generate 50,000 sets for the AM data, 30,400 for the FKM data, and 37,200 for the CFGK data.

Figure 1.(a) shows the distribution of the HE for the AM data and the random choice sets for those 35.92% of subjects who do not satisfy PHARP. The correlation of the HE and the AEI is $\rho_{\mathrm{AM}} = .615$. The mean HE is 0.955 for all, and 0.877 for those subjects who violate PHARP; 86.62% (73.24%) of the subjects have a HE greater than .881 (.957), while this is only the case for 5% (1%) of the random choice sets. The data of AM were also used to analyze differences between male and female subjects (see Andreoni and Vesterlund 2001); therefore data on the sex of a subject are available. The supplementary material includes a more detailed analysis of gender differences; it is found that female subjects are significantly more likely to violate PHARP (but not WGARP) than male subjects. The homothetic efficiency indices confirm that this difference is not merely due to minor violations.

Figure 1.(b) shows the same for the FKM data. The correlation of the HE and the AEI is $\rho_{\mathrm{FKM}} = .919$. The mean HE is .705; 86.84% (84.21%) of the subjects have a HE greater than .391 (.465), while this is only the case for 5% (1%) of the random choice sets.

The results for the risk experiment (CFGK) are shown in Figure 1. The correlation of the HE and the AEI is $\rho_{\mathrm{CFGK}} = .787$. The mean HE is .729.; 89.24% (84.95%) of the subjects have a HE greater than .447 (.521), while this is only the case for 5% (1%) of the random choice sets.

The supplementary material also shows the distribution of the $\widetilde{\mathrm{HE}}(i)$. Finally we would like to remark that while many subjects have very high HE levels compared to random choice, there is a point at which we have to ask if the *economic* significance of a deviation from homotheticity is not too great. Interpreted in economic terms, i.e. in terms of wasted income if preferences were homothetic, a deviation from 100% homotheticity may not be large compared to random choice, but economically, it can be quite substantial.

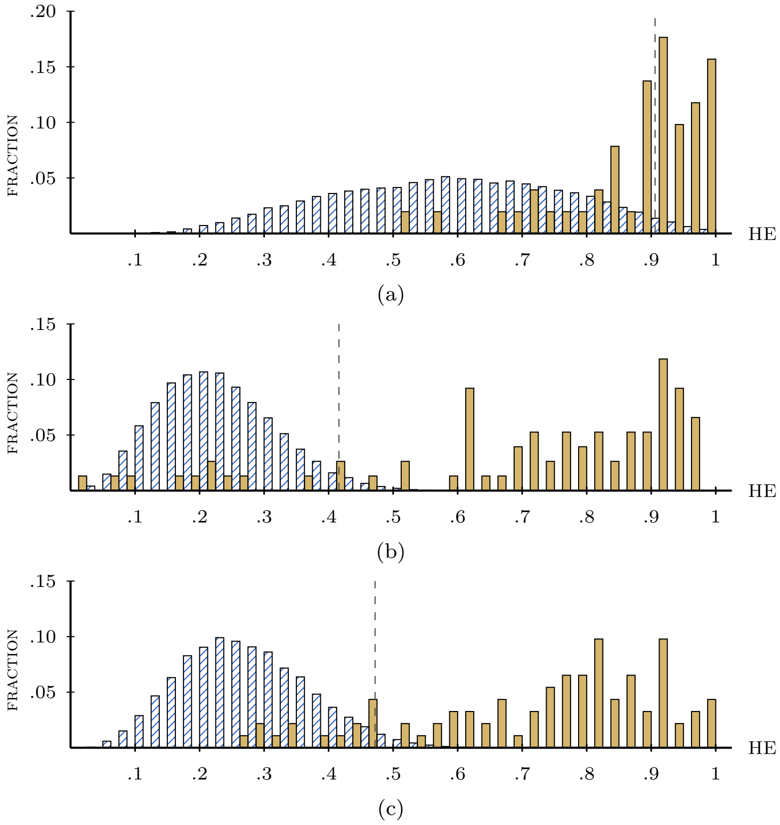

Fig. 1: [hatched] shows the distribution for random choices, [filled] for actual subjects. Distribution of the homothetic efficiency index and the Monte-Carlo results for the data of (a) Andreoni and Miller, (b) Fisman et al., (c) Choi et al. The dashed lines indicates the HE which exceeds the HE of 95% of the data sets in the Monte Carlo simulation.

5 Conclusion

In this article it was shown that for two dimensional commodity spaces any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set of observations. The result exploits the possibility of ranking budgets by their slope.

A straightforward application is to simplify the nonparametric test for homotheticity. Another useful application is to measure the extent of deviation from homotheticity. The described method was applied experimental data. We found that assuming homotheticity is justified for many subjects, but not for all. The procedure reveals for which subjects researchers should refrain from estimating parameters of homothetic utility functions.

A Appendix

Proof (Lemma 1) Choose a $\mathbf{z}^0$ without loss of generality. It is first shown that $\theta^{1,0} = t^{1,0}$. Remember that the observations are ordered such that $q^i \geq q^{i+1}$.

$$\theta^{1,0} = \frac{\mathbf{p}^1\mathbf{z}^0}{\mathbf{p}^1\mathbf{z}^1} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1}\frac{\mathbf{p}^i\mathbf{z}^0}{\mathbf{p}^i\mathbf{z}^i} \Leftrightarrow (\mathbf{p}^1\mathbf{z}^0)(\mathbf{p}^i\mathbf{z}^i) - (\mathbf{p}^1\mathbf{z}^i)(\mathbf{p}^i\mathbf{z}^0) \leq 0$$
$$\Leftrightarrow \quad (q^1 - q^i)(x^i y^0 - x^0 y^i) \leq 0.$$

The last line is true because if $i > 1$, the first term is positive and the second term is negative, and vice versa if $i < 1$. Now suppose

$$\theta^{1,0} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^0}{\mathbf{p}^k\mathbf{z}^k}$$

for a sequence $i, \ldots, k$ of length n. Then $\theta^{1,0}$ is also less than or equal to a sequence $i, \ldots, \ell$ of length $n+1$ because

$$\frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^0}{\mathbf{p}^k\mathbf{z}^k} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^\ell}{\mathbf{p}^k\mathbf{z}^k}\frac{\mathbf{p}^\ell\mathbf{z}^0}{\mathbf{p}^\ell\mathbf{z}^\ell}$$
$$\Leftrightarrow \quad (\mathbf{p}^k\mathbf{z}^0)(\mathbf{p}^\ell\mathbf{z}^\ell) \leq (\mathbf{p}^k\mathbf{z}^\ell)(\mathbf{p}^\ell\mathbf{z}^0),$$

where the last line is true for similar reasons as above. So $\theta^{1,0} = t^{1,0}$.

It is now possible to show that $\theta^{n,0} = t^{n,0}$ implies $\theta^{n+1,0} = t^{n+1,0}$, which concludes the proof by induction: Write $\theta^{n+1,0}$ as $\theta^{n+1,n}\theta^{n,0}$ and note that $\theta^{n+1,n} = (\mathbf{p}^{n+1}\mathbf{z}^n)/(\mathbf{p}^{n+1}\mathbf{z}^{n+1})$. Then it is to be shown that

$$\frac{(\mathbf{p}^{n+1}\mathbf{z}^n)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\theta^{n,0} \leq \frac{(\mathbf{p}^{n+1}\mathbf{z}^i)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\frac{(\mathbf{p}^i\mathbf{z}^j)}{(\mathbf{p}^i\mathbf{z}^i)} \cdots \frac{(\mathbf{p}^k\mathbf{z}^0)}{(\mathbf{p}^k\mathbf{z}^k)},$$

for sequences of $i, \ldots, k$ of arbitrary length. By assumption,

$$\theta^{n,0} \leq \frac{(\mathbf{p}^n\mathbf{z}^i)}{(\mathbf{p}^n\mathbf{z}^n)}\frac{(\mathbf{p}^i\mathbf{z}^j)}{(\mathbf{p}^i\mathbf{z}^i)} \cdots \frac{(\mathbf{p}^k\mathbf{z}^0)}{(\mathbf{p}^k\mathbf{z}^k)}.$$

It is then sufficient that

$$\frac{(\mathbf{p}^{n+1}\mathbf{z}^n)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})} \leq \frac{(\mathbf{p}^{n+1}\mathbf{z}^i)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\frac{(\mathbf{p}^n\mathbf{z}^n)}{(\mathbf{p}^n\mathbf{z}^i)}$$

holds, which is true if $n > 0$. The proof works analogously for $n < 0$.

Proof (Theorem 1) For (1) $\Leftrightarrow$ (2), see Varian (1983). It is obvious that (2) $\Rightarrow$ (3). We will show that (3) $\Rightarrow$ (2). Choose a $\mathbf{z}^0$ without loss of generality, i.e. assign indices such that q^0 is the highest, the lowest, or somewhere between the highest and the lowest relative price. Then $\left[\text{not } \mathbf{z}^0 P^0 \theta^{1,0}\mathbf{z}^1\right]$ if PHARP is satisfied. We need to show that this implies $\left[\text{not } \mathbf{z}^0 P^0 \theta^{n,0}\mathbf{z}^n\right]$ for all $n > 0$. So suppose $\left[\text{not } \mathbf{z}^0 P^0 \theta^{n,0}\mathbf{z}^n\right]$. Then

$$\mathbf{p}^0\mathbf{z}^0 \leq (\mathbf{p}^0\mathbf{z}^n)\theta^{n,0} \leq (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,0} = (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,n}\theta^{n,0}$$
$$\Leftrightarrow \quad \mathbf{p}^0\mathbf{z}^n \leq (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,n} \Leftrightarrow (\mathbf{p}^0\mathbf{z}^n)(\mathbf{p}^{n+1}\mathbf{z}^{n+1}) \leq (\mathbf{p}^0\mathbf{z}^{n+1})(\mathbf{p}^{n+1}\mathbf{z}^n)$$
$$\Leftrightarrow \quad (q^0 - q^{n+1})(x^{n+1}y^n - x^n y^{n+1}) \leq 0.$$

It is easy to see that the last line is true because if $n > 0$ and PHARP is satisfied the first term on the left hand side is positive while the second term is negative. A similar argument applies when $n < 0$. This proves that $\left[\text{not } \mathbf{z}^0 P^0 \theta^{1,0} \mathbf{z}^1\right]$ implies $\left[\text{not } \mathbf{z}^0 P^0 \theta^{n,0} \mathbf{z}^n\right]$ for arbitrary $\mathbf{z}^0$. So PHARP implies HARP.

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